The Problem of the Infinite Exponential

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To the memory of my Parents

Contents

1 Introduction **5**

2 An application of function W to infinite exponentials **6**
- 2.1 Definitions . . . 6
- 2.2 The W function . . . 7
- 2.3 The central lemma . . . 10
- 2.4 Convergence for $x \in \mathbb{R}$ . . . 11
- 2.5 Convergence for $c \in \mathbb{C}$ . . . 14
- 2.6 Convergence for $q \in \mathbb{Q}$ . . . 15

3 An application of functions HW to infinite exponentials **16**
- 3.1 Definitions . . . 16
- 3.2 The central lemma on HW . . . 17

4 Topology of the infinite exponential **18**
- 4.1 Julia and Fatou sets . . . 18
- 4.2 Attractors and repellers . . . 19
- 4.3 General topological map of the infinite exponential . . . 20
- 4.4 Topology . . . 22
- 4.5 Fractal nature in the infinite exponential . . . 22
- 4.6 The primitive of a Cantor bouquet . . . 26
- 4.7 Pinball with the infinite exponential . . . 27

5 An indexing scheme for the HW maps **27**
- 5.1 An algebraic scheme . . . 27
- 5.2 Polynomial functions . . . 28
- 5.3 Rational functions . . . 30
- 5.4 Analytic functions . . . 31
- 5.5 HW functional index . . . 33

6 The generalized infinite exponential **33**
- 6.1 Definitions . . . 33
- 6.2 Ascending indexes . . . 35
- 6.3 Descending indexes . . . 35

7 Appendix: programming/results **37**

8 Acknowledgements **39**

9 Bibliography **40**

1 Introduction

The problem of the infinite exponential was observed and studied for the first time by Leonhard Euler ([85]). It concerns the sequence defined recursively as follows:

Definition 1.1 *Given $x > 0$,*

$$\alpha_{n+1} = x^{\alpha_n}$$

The central problem of the infinite exponential is, then, the Proposition:

Proposition 1.2 *Given $x > 0$, study the behavior of the recursive sequence α_n, $n \in \mathbb{N}$.*

Euler proved the following theorem:

Theorem 1.3 *The sequence α_n converges if and only if $x \in \left[e^{-e}, e^{e^{-1}}\right]$.*

The central problem leads to an extensive labyrinth of sub-problems, which are examined partially or fully in the bibliography references. The reason why the area of sub-problems is deep and extensive, is that the sequence (1.1) is based on the arithmetic operator of exponentiation, which is the most “powerful” of the three $\{+, \times, \uparrow\}$, but unfortunately it is the only one that is not commutative. In other words, in general[1], $a^b \neq b^a$.

A second reason the central problem leads to a plethora of sub-problems, is that the two previous operators do not exhibit any idiopathic behavior relative to their repeated application. For example, for given $x > 1$, the repeated operations $+(n)x = \underbrace{x + x + \ldots + x}_{n}$ and $\times(n)x = \underbrace{x \times x \times \ldots \times x}_{n}$, cause divergence. The latter does not always happen with the exponentiation operator, as the sequence (1.1), may converge as with Euler's Theorem, for $x \in \left[e^{-e}, e^{e^{-1}}\right]$.

The above difference becomes immediately obvious for example, when a student attempts to investigate the behavior of the sequence (1.1), for given $x > 1$, let's say for $x = 1.2$, where a simple pocket calculator shows that the repeated exponentiation of such x leads to an unexpected convergence.

This unexpected convergence is the exceptionally strange of theorem (1.3), something which was observed by Euler, using only paper and pencil. Something like that is really inconceivable without the use of computers, at least insofar modern calculation methods are concerned, as the exponentiation operator exhibits mainly calculation difficulties which do not exist when using the other two previous operators, since repeated addition and multiplication are relatively “fast” operations when using computers.

We, then have a problem which could possibly be investigated further using fairly strong computer systems, which could conceivably help with many of the sub-problems that show up.

The first important development came around 1999-2000, when fairly advanced versions of the Computer Algebra Systems (CAS) Maple and Mathematica showed up [154].

[1] The problem of solving the equation $a^b \neq b^a$, which is thus of substantial importance in the investigation of the infinite exponential, was presented for the first time to the author in 1982, in a letter of the author's Father to the author

These packages include since then symbolic versions of the complex function W, otherwise known as Lambert's function. This latter function has found multiple applications the last ten to fifteen years in various sectors of pure and applied mathematics, like physics, electronics, optics and astronomy.

The function W is a limited inverse of the complex function $z \cdot e^z$, with the latter being intrinsically connected to the operator of the repeated exponentiation, because the last one can be analyzed in parts by $z \cdot e^z$, when one considers the partial sub-exponents of a repeated exponential, using the primitive base exp.

Finally and after years of research in arithmetical analysis of the results of the survey related to the infinite exponential and the Maple CAS Maple ([67, 332]), it is validated that the function W, gives a closed form for the convergence limit of the sequence (1.1) as it follows from Theorem (1.3), when the sequence (1.1) converges.

Therefore the function W becomes the ideal tool to examine the infinite exponential with and consequently allows a direct extraction of final conclusions related to the area of convergence of the sequence (1.1), extending Theorem (1.3) to the complex plane, to an area called today area Shell-Thron.

The above results are generalized directly to the space of Quaternions, with the new results coinciding with the already known ones for the complex plane, despite the Quaternion space not being (in general) commutative.

The problem of the infinite exponential does not end there, as arithmetic calculations also suggest non-convergence or even divergence of the sequence in many sub-areas of the complex plane, which need additional calculations in order to find out if in these areas the sequence (1.1) behaves normally and how.

Examining these areas of unknown behavior of the infinite exponential, requires the use of functions which are gradually more complex in nature than W and the author calls them HW. These functions solve additionally the cases where the sequence (1.1) falls into cycles of period $p > 1$, characterizing therefore fully the sub-areas of convergence of the sequence, when there are multiple attractors, which are given in closed form by the functions HW, and determine therefore the partial limits of the convergence.

This work collects sub-results until Proposition (1.2) of Euler is answered in its general form, which is essentially the topology of the sequence $\alpha_{n+1} = \gamma_n^{\alpha_n}$, for general $\gamma_n \in \mathbb{C}$, calculates a dimension bound for its main feature set and examines partially the behavior of the functions HW which allow the extraction of conclusions for this Proposition.

2 An application of function W to infinite exponentials

2.1 Definitions

The function W acquired substantial publicity lately, mainly because of important progress in computational mathematics. Even though compositions of it appear in hidden form in references [18, 153], [187, 14] and [122, 235], its essential properties are presented in [67, 344-349] and [68, 2-8]. Some of these properties can be used to simplify the answer of when the infinite exponential converges.

We are working with the principal branch of the complex map log, and we use Maurer's notation for repeated and infinite exponential (see [122, 239-240]).

It is assumed that equations with complex exponents anywhere in this dissertation use always the principal branch of the complex exponentiation, whenever this is needed: $c^w = e^{w \cdot \log(c)}$, $c \neq 0$, with log always denoting the principal branch of the complex logarithm $\log(k, z)$, for $\theta \in (-\pi, \pi]$.

Definition 2.1 *For $z \in \mathbb{C} \setminus \{x \in \mathbb{R} \colon x \leq 0\}$ and $n \in \mathbb{N}$,*

$$^{n}z = \begin{cases} z & , \textit{ if } n = 1, \\ z^{\left(^{n-1}z\right)} & , \textit{ if } n > 1. \end{cases}$$

Whenever the following limit exists and is finite, we set:

$$^{\infty}z = \lim_{n \to \infty} {}^{n}z \tag{1}$$

We use the exponential map $g_c(z)$ and its compositions, for $c \in \mathbb{C} \setminus \{x \in \mathbb{R} \colon x \leq 0\}$. Whenever the parameter c is obvious, it will be omitted to avoid any confusion.

$$g_c(z) = c^z \tag{2}$$

$$g_c^{(n)}(z) = \begin{cases} g_c(z) & \text{if } n = 1, \\ g_c\left(g_c^{(n-1)}(z)\right) & \text{if } n > 1. \end{cases} \tag{3}$$

^{n}z and $g^{(n)}(z)$ are related: $^{n}c = g_c^{(n)}(1)$. We also use the following function, which is a partial inverse of W:

$$m(z) = z \cdot e^z, z \in \mathbb{C} \tag{4}$$

The terminology *Infinite Exponential* seems to have been used first in [18, 150]. In short, it is the infinite tower $z_1^{z_2^{z_3^{\cdot^{\cdot^{\cdot}}}}}$, with $z_n \in \mathbb{R}$ (or $z_n \in \mathbb{C}$), $\forall n \in \mathbb{N}$. We are initially interested in the case $z_n = z$, $\forall n \in \mathbb{N}$. Respectively, the terminology $z^{z^{\cdots^{z}}}$ (real or complex) will be used alternatively to to the notation $z^{\left(z^{(\cdots z)}\right)}$ to notate repeated exponentiation from the top to the bottom. In the majority of the cases we use Definition (2.1) for the repeated exponential.

2.2 The W function

The complex function W is a partial inverse of $m(z)$ or otherwise the function which solves the equation $m(z) = w$ relative to z. Alternatively,

Definition 2.2 W *satisfies the functional equation:*

$$\mathrm{W}(z)e^{\mathrm{W}(z)} = z,\ z \in \mathbb{C}$$

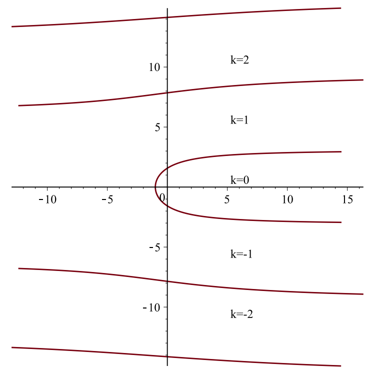

Figure 1: Quadratrix of Hippias bounds the ranges of $\mathrm{W}(k, z)$, $k \in \mathbb{Z}$

W is multivalued and has infinitely many branches as a complex Riemann surface. It is usually notated as $\mathrm{W}(k, z)$, with $k \in \mathbb{Z}$ specifying the working branch. Specifically, its principal branch $\mathrm{W}(0, z)$ corresponds to $k = 0$, and then the function is notated as $\mathrm{W}(z)$.

Follow some useful properties of W. Most of these can be found in [67] and [68] and can be validated numerically using the Maple CAS in [154, 305]. We selectively prove only some of the properties which are not immediately obvious in [68] or [67].

For $k \in \mathbb{Z}$, the various branches of $\mathrm{W}(k, z)$ are defined in $\mathbb{C}$ and are discontinuous at the points of the intervals BC_k:

$$BC_k = \begin{cases} \left(-\infty, -e^{-1}\right) & \text{, if } k = 0 \\ \left(-\infty, -e^{-1}\right) \cup \left(-e^{-1}, 0\right) & \text{, if } k = -1 \\ (-\infty, 0) & \text{, otherwise} \end{cases}$$

We observe that the branch point $z_0 = -e^{-1}$ of $\mathrm{W}(z)$ is $m(z)$, with z satisfying: $\frac{dm}{dz} = 0$. This branch point is shared between the two branches $\mathrm{W}(z)$ and $\mathrm{W}(-1, z)$. Suppose then that $CN = (-\infty, -1)$ and let us consider the curves:

$$C_k = \begin{cases} -y\cot(y) + yi, y \in (2k\pi, (2k+1)\pi), \text{ if } k \geq 0, \\ -y\cot(y) + yi, y \in ((2k+1)\pi, (2k+2)\pi), \text{ if } k < 0. \end{cases}$$

Lemma 2.3 *The image of BC_k under $\mathrm{W}(k, z)$ is:*

$$\mathrm{W}(k, BC_k) = \begin{cases} C_{-1} \cup CN, \textit{ if } k = -1 \\ C_k, \textit{ otherwise} \end{cases}$$

We now define the regions D_k as follows:

$$D_k = \begin{cases} \text{area between } C_1, CN, C_0, \text{ if } k = 1 \\ \text{area between } C_{-1}, CN, C_{-2}, \text{ if } k = -1 \\ \text{area between } C_k, C_{k-1}, \text{ otherwise} \end{cases}$$

The bounding curves of the ranges of $\mathrm{W}(k,z)$ C_k and the regions D_k give the well known quadratrix of Hippias, which is presented in [68] and in Figure 1.

The range of the images of W is then constrained by the regions D_k (see [67, 13-23]), consequently,

Lemma 2.4 $\mathrm{W}(k,\mathbb{C}\setminus\{0\})=D_k\cup C_k,\ k\in\mathbb{Z}\setminus\{0\},$ *and* $\mathrm{W}(0,\mathbb{C})=D_0\cup C_0$

$\mathrm{W}(k,z)$ is a partial inverse of $m(z)$ in the aforementioned regions $D_k\cup C_k$, consequently (see [67])

Lemma 2.5 $\mathrm{W}(k,m(z))=z,\ k\in\mathbb{Z},\ z\in D_k$ *and* $m(\mathrm{W}(k,z))=z,\ k\in\mathbb{Z},\ z\in\mathbb{C}$

Corollary 2.6 $\mathrm{W}(m(z))=z,\ z\in D_0$ *and* $m(\mathrm{W}(z))=z,\ z\in\mathbb{C}$

From the symmetry between C_k and D_k follows:

Lemma 2.7 $\overline{\mathrm{W}(k,z)}=\mathrm{W}(-k,\overline{z}),\ k\in\mathbb{Z},\ z\in\mathbb{C}$

Corollary 2.8 $\overline{\mathrm{W}(z)}=\mathrm{W}(\overline{z}),\ z\in\mathbb{C}$

Since $(-\infty,-1]\in D_{-1}$ and $[-1,+\infty)\in D_0$, only the branches that correspond to $k=0$ and $k=-1$ can ever assume real values.

Lemma 2.9 $\mathrm{W}(k,z)\in\mathbb{R}\Rightarrow k\in\{-1,0\}$

From the first diagram in [67] (or elementary calculus), follow,

Lemma 2.10 $\mathrm{W}(x)$ *is real, continuous and monotone increasing on the interval* $[-e^{-1},+\infty)$.

Lemma 2.11 $\mathrm{W}(-1,x)$ *is real, continuous and monotone decreasing on the interval* $[-e^{-1},0)$.

Lemmas (2.10) and (2.11) follow easily by considering the function $m(z)$ and the fact that $-e^{-1}$ is a common branch point between $\mathrm{W}(z)$ and $\mathrm{W}(-1,z)$:

Lemma 2.12 $\mathrm{W}(e)=1$

Lemma 2.13 $\mathrm{W}\left(-e^{-1}\right)=\mathrm{W}\left(-1,-e^{-1}\right)=-1$

Lemma 2.14 $D\subset D_0$ *and* $\partial D\cap\partial D_0=\{-1\}$

Proof: It suffices to show $|-y\cot(y)+yi|>1$ for each $y\in\left(0,\frac{\pi}{2}\right)$. We observe immediately that $\lim\limits_{y\to0^+}(-y\cot(y)+yi)=-1\in\partial D\cap\partial D_0$ and $z\in\partial D\cap\partial D_0\Rightarrow$ $\cos(\pi-y)+\sin(\pi-y)i=-y\cot(y)+yi\Rightarrow\{\sin(y)=y,-y\cot(y)=\cos(\pi-y)\}$. From the first equation we get $y=k\pi,\ k\in\mathbb{Z}$. From it only the equation $y=(2k+1)\pi,\ k\in\mathbb{Z}$ satisfies the second equation as a limit, therefore $z=\cos((2k+1)\pi)+\sin((2k+1)\pi)i=-1$ and the Lemma follows. $\square$

Lemma 2.15 $\mathrm{W}(z)$ *is analytic at the origin* $z_0 = 0$ *with series:*

$$S(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1} z^n}{n!}$$

and convergence radius: $R_s = e^{-1}$

Proof: Details related to the series $S(z)$ as well as other expansions are given in [67]. The Ratio test reveals the radius of convergence. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\left(1+\frac{1}{n}\right)^{n-1} z\right| = |ez| < 1$, or equivalently, $|z| < e^{-1}$. $S(z)$ is valid on the entire disk $D_w = \left\{z : |z| \leq e^{-1}\right\}$. When $|z| = e^{-1}$, then $\left|\frac{(-n)^{n-1} z^n}{n!}\right| = \frac{n^{n-1}}{e^n n!} < \frac{n^{n-1}}{\sqrt{2\pi n}^{\left(n+\frac{1}{2}\right)}} = \frac{1}{\sqrt{2\pi} n^{\frac{3}{2}}}$, using Stirling's approximation and the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi} n^{\frac{3}{2}}}$ converges. □

The fact that $S(z)$ has radius of convergence $R_s = e^{-1}$, follows also from the fact that $\mathrm{W}(z)$ has a branch point at $z_0 = -e^{-1}$.

2.3 The central lemma

the main result of Lemma (2.18) which concerns the limit of the Euler sequence is mentioned in [67, 332]. The author thinks it deserves a deeper analysis, particularly after the fact that W is a multivalued function.

$g(z)$ of (2) is intricately connected with the infinite exponential. Generally speaking, given $c \in \mathbb{C}$, $c \notin \{0, 1\}$, if the sequence $g^{(n)}(z)$, $n \in \mathbb{N}$ converges, then it must converge to a fixed point of $g(z)$ or equivalently the limit must satisfy the first auxiliary equation,

$$z = g(z) \tag{5}$$

Equation (5) can always be solved through W.

Lemma 2.16 *The fixed points of* $g(z)$ *are given by* $h\colon \mathbb{Z} \times \mathbb{C} \mapsto \mathbb{C}$, *with:*

$$h(k, c) = \frac{\mathrm{W}(k, -\log(c))}{-\log(c)},\ k \in \mathbb{Z}$$

Proof: $z = g(z) \Leftrightarrow z = c^z \Leftrightarrow ze^{-z\log(c)} = 1 \Leftrightarrow -z\log(c)e^{-z\log(c)} = -\log(c) \Leftrightarrow m(-z\log(c)) = -\log(c) \Leftrightarrow -z\log(c) = \mathrm{W}(k, -\log(c))$, $k \in \mathbb{Z}$, from the Definition of $m(z)$, $\Leftrightarrow z = \frac{\mathrm{W}(k,-\log(c))}{-\log(c)}$, $k \in \mathbb{Z}$ and the Lemma follows. □

Lemma 2.17 *If* $c \in \mathbb{C} \setminus \left\{e^{e^{-1}}\right\}$, *then* $h(k, c)$ *is a repeller of* $g(z)$.

Proof: If $c \neq e^{e^{-1}}$, then $g'\left(h(k, c)\right) = \log(c)\frac{\mathrm{W}(k,-\log(c))}{-\log(c)} = -\mathrm{W}(k, -\log(c)) \in -D_k$, from Lemma (2.4), therefore if $k \neq 0$, then $|g'\left(h(k, c)\right)| > 1$, from Lemma (2.12), and the Lemma follows. □

If $c = e^{e^{-1}}$, then $g'(h(-1, c) = -\mathrm{W}(-1, -e^{-1}) = -1$ (Lemma (2.11)).

The supposition $c \neq e^{e^{-1}}$ is important. Otherwise, $g'\left(h(k, c)\right) =-\mathrm{W}\left(k, -e^{-1}\right) = 1$, for $k = -1$ (Lemma (2.17)).

Lemmas (2.16) and (2.17)lead to the central Lemma of the dissertation:

Lemma 2.18 ***(Corless)*** *Whenever the limit of $g^{(n)}(z)$ exists finitely, its value is given by:*

$$h(c) = h(0,c) = \frac{\mathrm{W}(-\log(c))}{-\log(c)}$$

therefore

$$\lim_{n\to\infty} g^{(n)}(z) = \lim_{n\to\infty} g^{(n)}(1) = {}^{\infty}c = h(c)$$

2.4 Convergence for $x \in \mathbb{R}$

The fact that the repeated iterated exponential with base $\sqrt{2}$ converges to 2 can be validated arithmetically in [70, 70] and [117, 66] and is explained analytically in [171, 434], [127, 77] and [139, 643-646]. Considering the following relation for the positive square root of 2, then $(\sqrt{2})^2 = 2$. replacing the exponent of the relation with the left side of the equation, we get the sequence $g^{(n)}(2)$, $n \in \mathbb{N}$. From the previous Lemma when the limit exists, $\lim_{n\to\infty} g^{(n)}(z) = 2$, $n \in \mathbb{N}$ for suitable z, consequently,

$$^{\infty}(\sqrt{2}) = 2 \tag{6}$$

This will also be shown with the last Lemma of this section.

Lemma 2.19 *If $x \in \left[e^{-1}, e\right]$, then $h\left(x^{x^{-1}}\right) = x$.*

Proof: By definition, $h(y)$ solves the equation $x = y^x$ which is equivalent to $x^{x^{-1}} = y$, therefore it is a partial inverse of $y(x) = x^{x^{-1}}$. In the given interval $y(x)$ is 1-1 and onto the range $\left[e^{-1}, e\right]$, and the Lemma follows. □

Lemma 2.20 *If $x \in (e, +\infty)$, then $h\left(x^{x^{-1}}\right) = w \in (1, e)$, with $w^{w^{-1}} = x^{x^{-1}}$.*

Proof: $y(x)$ is continuous on $(1, +\infty)$, acquires a maximum at $x = e$, and $\lim\limits_{x\to\infty} y(x) = 1$, therefore there exists unique w in $(1, e)$, such that $w^{w^{-1}} = x^{x^{-1}}$, and the Lemma follows from lemma (2.19). □

Example: $y = 1.3304 \doteq 1.562^{\left(\frac{1}{1.562}\right)} \doteq 6.620^{\left(\frac{1}{6.620}\right)}$. Such values are given in closed form using W, but can also be given arithmetically or using other methods in the references which deal with the solution of the equation $x^y = y^x$, like [43, 763], [55, 222-226], [56, 78-83], [89, 137], [141, 233-237], [161, 316], [169, 444-447], [183, 141], and [125, 96-99]. In the references the authors observe that $y(x)$ is a partial inverse of $h(x)$, but don't define h through W.

Lemmas (2.19) and (2.20) in short:

Lemma 2.21

$$h\left(x^{x^{-1}}\right) = \begin{cases} x, & \text{if } x \in \left[e^{-1}, e\right]; \\ w, & w \in (1,e) : w^{w^{-1}} = x^{x^{-1}}, \ \text{if } x \in (e, +\infty). \end{cases}$$

Lemma 2.22 *If* $c = e^{-e}$, $x_0 = \log\left(\mathrm{W}\left(\log(c)^{-1}\right)\log(c)^{-1}\right)\log(c)^{-1}$, *and* $u(x) = g^{(2)}(x) - x$, *then,*

$$
\begin{aligned}
x_0 & \textit{ is the only critical point of } u(x), \textit{ in } [0,1] && (7a)\\
x_0 &= e^{-1} && (7b)\\
u(x_0) &= 0 && (7c)\\
\left(\frac{du}{dx}\right)_{x_0} &= 0 && (7d)\\
\frac{du}{dx} &< 0,\ x \in [0,1] - \{x_0\} && (7e)
\end{aligned}
$$

Proof: $\frac{du}{dx} = 0$ can be solved exactly through W. If $\frac{du}{dx} = 0$, then $g^{(2)}(x)g(x)\log(c)^2 = 1$, therefore $e^{y\log(c)}y\log(c) = \log(c)^{-1}$, where $y = c^x$. Then $m(y\log(c)) = \log(c)^{-1}$, therefore $y\log(c) = \mathrm{W}\left(k, \log(c)^{-1}\right)$, consequently $y = \mathrm{W}\left(k, \log(c)^{-1}\right)\log(c)^{-1}$, and finally $x = \log\left(\mathrm{W}\left(k, \log(c)^{-1}\right)\log(c)^{-1}\right)\log(c)^{-1}$, $k \in \mathbb{Z}$. (7a) follows directly from Lemma (2.5). (7b) follows from Lemma (2.11). (7c) and (7d) follow easily. For (7e) we observe that $\log(c) = -e < 0$, therefore, if $x < x_0$ then $g(x) > e^{-1}$ and $g^{(2)}(x) < e^{-1}$, therefore $g^{(2)}(x)g(x)\log(c)^2 < 1$, consequently $\frac{du}{dx} < 0$. For $x > x_0$ the proof (with inequality signs reversed) is similar and the Lemma follows. □

Lemma 2.23 *If* $c \in \left\{e^{-e}, e^{e^{-1}}\right\}$, *then* ${}^{\infty}c = h(c)$.

Proof: If $c = e^{-e}$, the fixed point of $g(x)$ is given by Lemma (2.18). $h(c) = h\left(e^{-e}\right) = \frac{\mathrm{W}\left(-\log\left(e^{-e}\right)\right)}{-\log\left(e^{-e}\right)} = \frac{\mathrm{W}(e)}{e} = e^{-1}$, based on Lemma (2.26). Using Lemma (2.22), the continuity of $u(x)$ and the factoids: $u(0) = c > 0$, $u(1) = c^c - 1 < 0$, it follows that $g^{(2)}(x) > x$, if $x \in \left(0, e^{-1}\right)$ and $g^{(2)}(x) < x$, if $x \in \left[e^{-1}, 1\right]$. Using the two inequalities and induction on n, the sequence: $a_n = g^{(n)}(1)$, $n \in \mathbb{N}$ satisfies, $a_{2n+2} < a_{2n}$, and $a_{2n+3} > a_{2n+1}$, for each $n \in \mathbb{N}$. The last shows that a_{2n+1} and a_{2n} are monotone increasing and decreasing, respectively. Additionally, since $0 < c = e^{-e} < 1$, both sequences are bounded above by 1 and below by 0. It follows that both sequences a_{2n+1} and a_{2n} possess limits. Since the only root of $u(x)$ is x_0 (otherwise equation (7a) is violated), both sequences converge to x_0, from which follows that a_n converges to $x_0 = e^{-1}$.

If $c = e^{e^{-1}}$, then the fixed point of $g(x)$ is given again by Lemma (2.18). $h(c) = h\left(e^{e^{-1}}\right) = \frac{\mathrm{W}\left(-\log\left(e^{e^{-1}}\right)\right)}{-\log\left(e^{e^{-1}}\right)} = \frac{\mathrm{W}\left(-e^{-1}\right)}{-e^{-1}} = e$, based on Lemma (2.27). Using induction on n, the sequence $a_n = g^{(n)}(1)$, $n \in \mathbb{N}$ is monotone increasing and bounded above by e, therefore it converges to e and the Lemma follows. □

[122, 240], [18, 153] and [187, 14-15] arrive at the same result differently, without using the function W.

Lemma 2.24 *If* $c \in (0, e^{-e})$, *then* ${}^{\infty}c$ *does not exist.*

Proof: The fixed point $h(c)$ of $g(x)$ from Lemma (2.18) is a repeller. If $c \in (0, e^{-e})$, then $\mathrm{W}(e) < \mathrm{W}(-\log(c))$ based on Lemma (2.10) and consequently $1 < \mathrm{W}(-\log(c))$, based on Lemma (2.26). This means $1 < |g'(h(c))|$ and the Lemma follows from fixed point iteration. □

Lemma 2.25 *If* $c \in \left(e^{-e}, e^{e^{-1}}\right)$*, then* ${}^{\infty}c = h(c)$*.*

Proof: The fixed point $h(c)$ of $g(x)$ from Lemma (2.18) is an attractor. If $c \in \left(e^{-e}, e^{e^{-1}}\right)$, then $-e^{-1} < -\log(c) < e$, therefore $\mathrm{W}\left(-e^{-1}\right) < \mathrm{W}(-\log(c)) < \mathrm{W}(e)$, based on Lemma (2.7) and consequently $|g'(h(c))| < 1$ based on Lemmas (2.10) and (2.11) and the Lemma follows from fixed point iteration. $\square$

Lemma 2.26 *If* $c \in \left(e^{e^{-1}}, +\infty\right)$*, then* ${}^{\infty}c$ *does not exist.*

Proof: The fixed point $h(c)$ of $g(x)$ from Lemma (2.18) is a repeller. If $c \in \left(e^{e^{-1}}, +\infty\right)$, then $-\log(c) \in BC_0$, therefore $\mathrm{W}(-\log(c)) \in C_0$, based on Lemma (2.12), and consequently $|g'(h(c))| > 1$, based on the same Lemma and the Lemma follows from fixed point iteration. $\square$

Lemmas (2.23)-(2.26) and (2.18) validate the final Lemma of this section which is Theorem (1.3) of Euler:

Lemma 2.27 ${}^{\infty}c$ *exists, then and only, when* $c \in \left[e^{-e}, e^{e^{-1}}\right]$*, me* ${}^{\infty}c = h(c)$*.*

Proof: The interval of convergence can be determined for the real case from fixed point iteration. The only potential fixed point of $g(x)$ is given from Lemma (2.18), as $h(c)$. Using elementary properties of all the relevant functions, if $|g'(h(c))| \leq 1$, then $|-\mathrm{W}(-\log(c))| \leq 1$. This means $\mathrm{W}(-\log(c)) \in [-1, 1]$, therefore $m\left(\mathrm{W}(-\log(c))\right) \in m\left([-1,1]\right)$, or $m\left(\mathrm{W}(-\log(c))\right) \in \left[-e^{-1}, e\right]$, therefore also $-\log(c) \in \left[-e^{-1}, e\right]$, based on Definition (2.2) and finally $c \in \left[e^{-e}, e^{e^{-1}}\right]$. $\square$

Using Lemma (2.27) for $c = e^{e^{-1}}$, $c = e^{-e}$ and $c = \sqrt{2}$,

$$
\begin{aligned}
{}^{\infty}\left(e^{e^{-1}}\right) = \left(e^{e^{-1}}\right)^{\left(e^{e^{-1}}\right)^{\cdots}} &= h\left(e^{e^{-1}}\right) = e \\
{}^{\infty}\left(e^{-e}\right) = \left(e^{-e}\right)^{\left(e^{-e}\right)^{\cdots}} &= h\left(e^{-e}\right) = e^{-1} \\
{}^{\infty}\left(\sqrt{2}\right) = \left(\sqrt{2}\right)^{\left(\sqrt{2}\right)^{\cdots}} &= h\left(2^{\frac{1}{2}}\right) = 2
\end{aligned}
$$

The last equations settle normally the question in the beginning of this section with equation (6). That the infinite exponential converges if and only iff its base belongs to the interval $\left[e^{-e}, e^{e^{-1}}\right] \doteq [0.06598, 1.44466]$ is also shown in [122, 240], [139, 645] and [146, 556] using other methods and without employing the function W.

[9, 207-208] and [135, 301-303] also show that for $k \in \mathbb{N}$, $\lim_{c \to 0^+} {}^{2k}c = 1$ and $\lim_{c \to 0^+} {}^{2k+1}c = 0$. If $c \in (0, e^{-e})$, then ${}^{n}c$, $n \in \mathbb{N}$ is a cycle of period 2 by considering the even and odd subsequences. The bifurcation that occurs and its behavior are analyzed in [9, 207], [187, 15] and [135, 299]. We note that the two branches which stem from the bifurcation point $\left\{e^{-e}, e^{-1}\right\}$ can be given in parametric form as $a^{\left(\frac{a}{1-a}\right)}$ and $a^{\left(\frac{1}{1-a}\right)}$ for proper positive a (see for example [122, 237] or [188, 212]). In this case, as is shown in [171, 434], [122, 241-243], [187, 13] and [129, 501], the two separate limits $a = \lim_{n\to\infty} {}^{2n+1}c$ and $b = \lim_{n\to\infty} {}^{2n}c$ satisfy the inequality $0 < a < h(c) < b < 1$ and the *second auxiliary system*,

$$\begin{cases} a &= c^{c^a} \\ b &= c^a \end{cases} \tag{8}$$

A closed form solution for system (8) is given in detail in a subsequent section, where the difficulties of solving the auxiliary equation of order n are presented, for the complex exponential map $g(z)$, using certain generalized functions based on W.

2.5 Convergence for $c \in \mathbb{C}$

Suppose D is the unit disk. We consider the map $\phi\colon \mathbb{C} \mapsto \mathbb{C}$, defined as: $\phi(z) = e^{\left(\frac{z}{e^z}\right)} = e^{-m(-z)}$. The image $R_{ST} = \phi(D)$ is a nephroid region called **Shell-Thron** region, with an approximation given in [209] and in Figure 2.

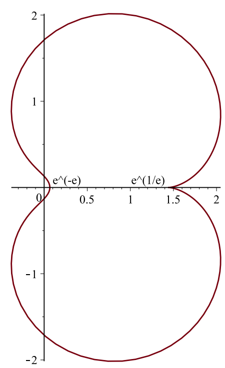

Figure 2: Shell-Thron region

[165, 679], [164, 12] and [13, 106] show that on the interior of R_{ST} we have convergence of $g^{(n)}(z)$. What happens at the boundary of the figure is mentioned in [12, 502] and [11]:

Theorem 2.28 ***(Baker/Rippon)*** *The sequence* ${}^nc$, $n \in \mathbb{N}$ *converges for* $\log(c) \in \{te^{-t} : |t| < 1,$ *or* $t^n = 1,$ *for some* $n \in \mathbb{N}\}$ *and diverges elsewhere.*

t and c are related through W, by considering always the principal branches of all maps involved. $c = \phi(t) \Leftrightarrow \mathrm{W}(-\log(c)) = \mathrm{W}(m(-t)) \Leftrightarrow t = -\mathrm{W}(-\log(c))$, using Corollary(2.6) and Lemma (2.12). Consequently ϕ is reversible and $\phi^{-1} = (-\mathrm{W}) \circ (-\log)$ with $t = \phi^{-1}(c)$. Then we have the Theorem:

Theorem 2.29 *The sequence* ${}^nc$, $n \in \mathbb{N}$ *converges if* $\left|\phi^{-1}(c)\right| < 1$ *or* $\left(\phi^{-1}(c)\right)^n = 1$, *where* $\phi^{-1} = -\mathrm{W}(-\log(c))$, *some* $n \in \mathbb{N}$ *and converges elsewhere.*

Lemma 2.30 *If* $c \in \mathbb{C}$, *then the multipler of the fixed point* $h(c)$ *of* $g(z)$ *is given by* $t = \phi^{-1}(c)$.

Proof:

$$\begin{aligned}
g'\left(h(c)\right) &= \log(c) \cdot g(h(c)) \\
&= \log(c) \cdot h(c) \\
&= \log(c) \cdot \frac{\mathrm{W}\left(-\log(c)\right)}{-\log(c)} \\
&= -\mathrm{W}(-\log(c)) = t
\end{aligned}$$

and the Lemma follows. □

Theorem (2.28) then alternatively states that when $c = \phi(t)$, then the sequence $g^{(n)}(c)$, $n \in \mathbb{N}$ converges if and only if the measure of the multiplier $t = \phi^{-1}(c)$ of the fixed point $h(c)$ ths $g(z)$ is less than 1, or the multiplier is a n-th root of unity.

Lemma 2.31 *If $|t| < 1$ and $c = \phi(t)$, then ${}^{\infty}c = h(c)$.*

Proof: The fixed point of $g(z)$ is given by Lemma (2.18), as $h(c)$. $h(c) = \frac{\mathrm{W}(-\log(c))}{-\log(c)} = \frac{\mathrm{W}(-te^{-t})}{-te^{-t}} = \frac{\mathrm{W}(m(-t))}{-te^{-t}} = \frac{-t}{-te^{-t}} = e^t$, using Lemma (2.4) and Corollary (2.6). The fixed point $h(c) = e^t$ of $g(z)$ is an attractor. According to Lemma (2.30), $|g'(h(c))| = |t| < 1$, which is true by assumption and the Lemma follows from fixed point iteration theory and Lemma (2.18). □

Lemma 2.32 *If $|t| > 1$ and $c = \phi(t)$, then ${}^{\infty}c$ does not exist.*

Proof: The fixed point $h(c) = e^t$ of $g(z)$ is a repeller: According to (2.30), $|g'(h(c))| = |t| > 1$, which is true by assumption and the Lemma follows similarly. □

More applications for this section, like Fractals based on repeated exponential iteration and more identities, may be verified in [209].

2.6 Convergence for $q \in \mathbb{Q}$

Generalizing the function $\mathrm{W} = W_c$ as W_h in the space $\mathbb{Q}$, we note that convergence still happens. First we get a generalization of the Theorem of Shell ([164]).

Lemma 2.33 *If $q \in \mathbb{Q}$, then the sequence ${}^{n}q$, $n \in \mathbb{N}$ converges if $t = \phi^{-1}(q) \in (S^3)^{\circ}$.*

Lemma 2.34 *If $q \in \mathbb{Q}$ and the sequence $g^{(n)}(q)$ converges, the value of the limit is always given as:*

$$h(q) = \frac{W_h(-\ln(q))}{-\ln(q)}$$

In the space of $\mathbb{Q}$ the function ln is not (in general) commutative, therefore we have two convergence orbits, however the orbits finally converge to the same point given by Lemma (2.34) which serves as a generalization of Lemma (2.18).

Examples: If $q = 1/2 - 1/2i + 1/2j - 1/2k$ then $|t_q| = |-W_H(-\log(q))| \doteq 0.70724 < 1$, therefore $t_q \in (S^3)^{\circ}$, therefore according to Lemma (2.33) the sequence ${}^{n}q$ converges. The first 14 values with Maple are given as:

$$
\begin{aligned}
{}^1q &\doteq 0.5 - 0.5i + 0.5j - 0.5k\\
{}^2q &\doteq 0.34967 - 0.11655i + 0.11655j - 0.11655k\\
{}^3q &\doteq 0.75577 - 0.16732i + 0.16732j - 0.16732k\\
{}^4q &\doteq 0.51884 - 0.30319i + 0.30319j - 0.30319k\\
{}^5q &\doteq 0.49389 - 0.17222i + 0.17222j - 0.17222k\\
{}^6q &\doteq 0.63600 - 0.20888i + 0.20888j - 0.20888k\\
{}^7q &\doteq 0.53831 - 0.24422i + 0.24422j - 0.24422k\\
{}^8q &\doteq 0.54276 - 0.19809i + 0.19809j - 0.19809k\\
{}^9q &\doteq 0.58839 - 0.21696i + 0.21696j - 0.21696k\\
{}^{10}q &\doteq 0.55060 - 0.22511i + 0.22511j - 0.22511k\\
{}^{11}q &\doteq 0.55731 - 0.20924i + 0.20924j - 0.20924k\\
{}^{12}q &\doteq 0.57094 - 0.21767i + 0.21767j - 0.21767k\\
{}^{13}q &\doteq 0.55692 - 0.21898i + 0.21898j - 0.21898k\\
{}^{14}q &\doteq 0.56109 - 0.21373i + 0.21373j - 0.21373k
\end{aligned}
$$

The central Lemma (2.34) gives the limit: ${}^{\infty}q = h(q) \doteq 0.56182 - 0.21640i + 0.21640j - 0.21640k$.

If $q = 1 + i + j - k$ then $|t_q| = |-W_H(-\log(c))| \doteq 0.94126 < 1$, therefore $t_q \in (S^3)^{\circ}$, consequently again based on Lemma (2.33) the sequence ${}^nq$ converges also. Lemma (2.34) gives the limit: ${}^{\infty}q = h(q) \doteq 0.46827 + 0.33791i + 0.33791j - 0.33791k$.

Details and extensive analysis for the results in the space $\mathbb{Q}$ are given in [206].

3 An application of functions HW to infinite exponentials

Divergence in the previous sections doesn't necessarily mean that $|{}^{\infty}c| = \infty$. Divergence may be such that the limit is ∞, but can also be periodic and even totally chaotic. The periodic points given by (2.18) all have period 1, so it follows that these are automatically solutions to the n-th auxiliary equation $g^{(n)}(z) = z$, but not all solutions are given only from the main Lemma. Such points exist. To determine such points we need functions which are stronger than W.

3.1 Definitions

Suppose $f_i(z)$ are non-vanishing identically complex functions. We define $F_{n,m}(z)\colon \mathbb{N}^2 \times \mathbb{C} \to \mathbb{C}$ as:

Definition 3.1

$$F_{n,m}(z) = \begin{cases} e^z & , \textit{ if } n = 1,\\ e^{c_{m-(n-1)}F_{n-1,m}(z)} & , \textit{ if } n > 1. \end{cases}$$

Definition 3.2 $G(f_1, f_2, \ldots, f_k; z) = z \cdot F_{k+1,k+1}(z)$

If $k = 0$, then $G(z) = z \cdot e^z$. If $k = 1$ then $G(c_1; z) = ze^{c_1 e^z}$. If $k = 2$ then $G(c_1, c_2; z) = ze^{c_1 e^{c_2 e^z}}$. When we write about the HW, we can use the terminology $G(\ldots; z)$, meaning that the corresponding function includes meaningful terms-parameters. The function of interest here is the inverse of $G(\ldots; z)$.

$$\mathrm{HW}(f_1, f_2, \ldots, f_k; y) \tag{9}$$

In other words G and HW satisfy the functional relation:

$$G(\ldots; \mathrm{HW}(\ldots; y)) = y \tag{10}$$

by supposing always that the list of parameters is identical on both sides. These maps will henceforth be called HW functions. We note that when $k = 0$, $\mathrm{HW}(y) = \mathrm{W}(y)$ is the function W. The existence of the HW is guaranteed in all cases by the Lagrange Inversion Theorem (see [159, 201-202]).

3.2 The central lemma on HW

A generalization is now immediate, so we are close to a variant of the central Lemma.

Lemma 3.3 *If $c \in \mathbb{C} \setminus \{0, 1\}$, then the p-th auxiliary equation $f^{(p)}(z) = z$, admits the solutions $\left\{z, f(z), \ldots, f^{(p-1)}(z)\right\}$, where*

$$z = \frac{\mathrm{HW}(-\log(c), \ldots, \log(c); \log(c))}{\log(c)} \text{ (p parameters)}$$

Proof: Similarly to the central Lemma, the functions HW solve the p-th equation $g^{(p)}(z) = z$ as follows:

$$\begin{aligned}
g^{(p)}(z) &= z \Rightarrow \\
c^{c^{\cdots^{c^z}}} &= z \Rightarrow \\
z \cdot c^{-c^{\cdots^{c^z}}} &= 1 \Rightarrow \\
z \cdot e^{-\log(c) e^{\cdots^{e^{\log(c) z}}}} &= 1 \Rightarrow \\
z \log(c) \cdot e^{-\log(c) e^{\cdots^{e^{\log(c) z}}}} &= \log(c) \Rightarrow \\
z \log(c) &= \mathrm{HW}(-\log(c), \ldots, \log(c); \log(c)) \Rightarrow \\
z &= \frac{\mathrm{HW}(-\log(c), \ldots, \log(c); \log(c))}{\log(c)} \text{ p parameters}
\end{aligned}$$

We note that if z is a solution of $g^{(p)}(z) = z$, then if $k \in \{1, 2, \ldots, p-1\}$ we also have $g^{(p)}(g^{(k)}(z)) = g^{(k)}(g^{(p)}(z)) = g^{(k)}(z)$ and the Lemma follows. $\square$

Corollary 3.4 *Whenever ${}^{n}c$, $n \in \mathbb{N}$ is an attracting cycle, the limits of the p subsequences ${}^{n+k}c$, $n \in \mathbb{N}$, $k \in \{0, 1, \ldots, p-1\}$ are given as $\left\{z, g(z), \ldots, g^{(p-1)}(z)\right\}$, with z being given by the expression of Lemma* (3.3).

Finally,

Lemma 3.5 *If* $c \in \mathbb{C} \setminus R_{ST}$ *and* $t = \mathrm{HW}(-\log(c), \ldots, \log(c); \log(c))$ *(p parameters), then the sequence* ${}^n c$*,* $n \in \mathbb{N}$ *is an attracting cycle of period* p*, if* $\left|te^t \prod_{k=1}^{p-2} f^{(k)}(e^t)\right| < \left|\log(c)^{-p+1}\right|$ *and only if* $\left|te^t \prod_{k=1}^{p-2} f^{(k)}(e^t)\right| \leq \left|\log(c)^{-p+1}\right|$.

Proof: The multiplier of the fixed point z of the function $g^{(n)}(z)$ is given in [208], as:

$$\left(g^{(n)}\right)'(z) = (\log(c))^n \cdot \prod_{k=1}^{n} g^{(k)}(z) \tag{11}$$

The fixed point z is given by the expression of Lemma (3.3), through the HW function, so substituting this point in equation (11), the multiplier takes the form:

$$(\log(c))^{p-1} \cdot te^t \prod_{k=1}^{p-2} g^{(k)}(e^t) \tag{12}$$

The attractor condition of the function $g^{(p)}(z)$ at the fixed point z is $\left|\left(g^{(n)}(z)\right)'\right| \leq 1$, therefore the Lemma follows from equation (12). □

Example: Can we say anything for example about the sequence ${}^n c$, $n \in \mathbb{N}$, if $c = -1+i$? We can use the code of the Appendix to check the measure of the corresponding multipliers t_p for p-cycles. $t_1 = \phi^{-1}(c) = -\mathrm{W}(-\log(c)) \doteq 1.13445$, therefore it cannot be a cycle of period 1 (in other words we don't have convergence). $t_2 \doteq 0.80847 + .33448i$, and $|t_2 e^{t_2} \log(c)| \doteq 4.67684$, therefore it cannot be a cycle of period 2. $t_3 \doteq 0.03281 - 0.08534i$ and $|t_3 e^{t_3} f(e^{t_3}) \log(c)^2| \doteq .94227$, therefore it has to be necessarily a cycle of period 3. From Corollary (3.4), $z = \mathrm{HW}(-\log(c), \log(c); \log(c))/\log(c) \doteq -0.03344 - 0.01884i$, therefore the three limits of the cycle are given as $\left\{z, g(z), g^{(2)}(z)\right\}$. This set is calculated based on the code with accuracy to 5 decimal digits as, $-0.03344 - 0.01884i$, $1.02959 - 0.08808i$, $-1.29112 + 1.19363i$. Convergence to these points is spiral like (see [208]).

4 Topology of the infinite exponential

4.1 Julia and Fatou sets

To study generally (and specifically) the behavior of the sequence of Euler, we study the corresponding topology which results on the complex plane from repeatedly iterating the exponential map, for different (but specific each time) base of the exponential, since Euler's sequence essentially reduces to repeated iteration of some suitable exponential map, for a specific base $\lambda = \log(c)$, with $c \in \mathbb{C}$ and initial value $z_0 \in \mathbb{C}$.

This last topology is the union of two notable sets of the exponential family, that is the Julia set $\mathfrak{J}[g_c]$ and the Fatou set $\mathfrak{F}[g_c]$ of the exponential, which are complementary on the complex plane.

To characterize therefore the topology of the sequence of Euler, it suffices to characterize exactly the sets $\mathfrak{J}[g_c]$ and $\mathfrak{F}[g_c]$ of the exponential map. This characterization has been done however in the references for the family of maps $E_\lambda(z) = \lambda \cdot e^z$, with $\lambda = \log(c)$.

Specifically, the Julia set of the exponential map $E_\lambda(z) = \lambda \cdot e^z$ is given in [76], [75] and [77] and is the complement of the set of periodic repellers of $E_\lambda(z)$, that is, $\mathfrak{J}[E_\lambda] = \left\{z \in \mathbb{C}\colon E_\lambda^{(n)}(z) \to \infty\right\}$. The Fatou set of the exponential map E_λ, is the complement of the set $\mathfrak{J}[E_\lambda]$ or the set of its periodic attractors, that is $\mathfrak{F}[E_\lambda] = \left\{z \in \mathbb{C}\colon E_\lambda^{(n)}(z) \leq M\right\}$.

Proposition 4.1 *The Julia and Fatou sets of the topology of the infinite exponential, coincide with the corresponding sets of the repeated iteration of the exponential map $E_\lambda(z) = \lambda \cdot e^z$, with $\lambda = \log(c)$.*

Proof: In the aforementioned references it is proved that the Julia set $\mathfrak{J}[E_\lambda]$ of the family of exponential maps $E_\lambda(z) = \lambda \cdot e^z$ is a *Cantor bouquet* for $0 < \lambda < 1/e$, therefore it suffices to show that in our case the set $\mathfrak{J}[g_c] = \mathfrak{J}[c^z]$ is in general such a bouquet. Note that the iteration orbit in the references is $E = \left\{\lambda \cdot e^z, \lambda \cdot e^{\lambda \cdot e^z}, \ldots\right\}$. This orbit can be written alternatively also as $E = \left\{\lambda \cdot e^z, \lambda \cdot (e^\lambda)^{e^z}, \ldots\right\}$, therefore setting $\lambda = \log(c)$, the orbit is equivalent to the orbit $E_{\lambda=\log(c)} = \left\{\log(c) \cdot e^z, \log(c) \cdot c^{e^z}, \ldots\right\}$. For $z = 0$, we get the orbit of zero, as $E_0 = \{\log(c), \log(c) \cdot c, \ldots\}$. We observe that $e^{E_0} = \left\{(1,)c, c^c, c^{c^c}, \ldots\right\}$, which is the simplest orbit of the Euler sequence for the infinite exponential, therefore the set $\mathfrak{J}[g]$ will be exactly what the set $\mathfrak{J}\left(E_{\lambda=\log(c)}\right)$ is, on the complex plane $\lambda = \log(c)$. Specifically, it is a Cantor bouquet. Since the orbits are coincident modulo λ, the same will hold also for the Fatou sets and the proof is complete. □

4.2 Attractors and repellers

We classify now the topology of specific sub-cases for given $\lambda = \log(c)$ in a neighborhood B_δ, $\delta > 0$ of the boundary of the Shell-Thron region, by period.

Proposition 4.2 *When for $\lambda = \log(c)$ the set $\mathfrak{J}[g]$ is a Cantor bouquet p-furcation, then, if there are attractors in the set $\mathfrak{F}[g]$, they are given from the expression $A_1 = \mathrm{W}(-\lambda)/(-\lambda)$ if $p = 1$ or from the expression $A_p = \left\{g^k(z_0), z_0 = \mathrm{HW}(\ldots, \lambda)/\lambda\right\}$ $k \in \{0, 1, \ldots, p-1\}$, if $p \geq 2$.*

Proof: The proof is done inductively depending on the value of $\lambda = \log(c)$ on the complex plane, meaning, depending on the position of $c = e^\lambda$ relative to the region R_{ST}.

- $c \in (R_{ST} \cap B_\delta(\phi(t)))^\circ$. Setting $\lambda = \log(c) = \log(\phi(t)) = t/e^t$, by assumption is $0 < |t| < 1$, therefore we get the sub-case $0 < |\lambda| = |t/e^t| < 1/e$. In this case, the sequence of the infinite exponential converges to $A_1 = h(0, c) = -\mathrm{W}(-\lambda)\lambda$ and the set $\mathfrak{J}[g_c(z)]$ is a Cantor bouquet. The set $\mathfrak{F}[g_c(z)]$ then is the complement of the set $\mathfrak{J}[g_c(z)]$ on the complex plane and contains exactly one attractor A_1, which drives any point of $\mathfrak{F}[g_c(z)]$ to A_1.

- $c \in \partial R_{ST} \cap B_\delta(\phi(t))$. Setting $\lambda = \log(c) = \log(\phi(t)) = t/e^t$, by assumption is $|t| = 1$, therefore we get the sub-case $|\lambda| = |t/e^t|$. In this case, when $p = 1$, then $t = 1$ and $|\lambda| = 1/e$. The set $\mathfrak{J}[g_c(z)]$ then is a plain Cantor bouquet displaying a single fork at the neutral point $N_1 = h(0, e^{e^{-1}}) = e$. The set $\mathfrak{F}[g_c(z)]$ is simply the complement of the set $\mathfrak{J}[g_c(z)]$ on the complex plane. When it happens $t^p = 1$ with $p = 2$, it also is $t = -1$ and the sequence of the infinite exponential breaks into a fork of period $p = 2$ on the complex plane, therefore the set $\mathfrak{J}[g_c(z)]$ consists of two Cantor bouquets displaying a double fork at the neutral point $N_2 = h(0, e^{-e}) = 1/e$. The set $\mathfrak{F}[g_c(z)]$ then, consists of $p = 2$ sub-regions without attractors. Continuing inductively, when it is $t^p = 1$, $p > 2$, the sequence breaks into a fork of period $p > 2$ at the neutral point $N_3 = h(0, c) = -\operatorname{W}(-\lambda)/\lambda$, therefore the set $\mathfrak{J}[g_c(z)]$ consists of two Cantor bouquets displaying a p-furcation at the neutral point N_3. The set $\mathfrak{F}[g_c(z)]$ then, consists of $p > 2$ sub-cases without attractors.

- $c \in [(R_{ST})^c \cap B_\delta(\phi(t))]^\circ$. Setting $\lambda = \log(c) = \log(\phi(t)) = t/e^t$, we get the sub-case $|\lambda| = |t/e^t|$ with $|t| > 1$ and $t^p = 1$. If $p \geq 2$, then the case reduces to sub-cases $p \geq 2$ of the cases 1 and 2, above. Consequently, if $p > 1$, the set $\mathfrak{J}[g_c(z)]$ consists of two Cantor bouquets displaying a p-furcation at the repeller $R_1 = -\operatorname{W}(\lambda)/\lambda$ and the set $\mathfrak{F}[g_c(z)]$ consists of p sub-areas of the complex plane (Fatou flower in [78]), with each petal of the flower containing the periodic attractor $A_p = \{g^{(k)}(z_0), z_0 = \operatorname{HW}(\ldots, \lambda)/\lambda\}$, $k \in \{0, 1, \ldots, p-1\}$, which leads any of the points in the petals to A_p.

$\phi(t)$, which delineates the region R_{ST}, is conformal at every point of the unit circle except at the point $\phi(1) = e^{e^{-1}}$, therefore the three cases above cover fully the boundary of R_{ST} except for a neighborhood $B_\delta(\phi(1))$, $\delta > 0$, of the point $\phi(1)$, at which the map ceases to be conformal. In any neighborhood of this point we can have $|\lambda| \geq 1/e$, and in such a case, the corresponding set $\mathfrak{J}[g_c(z)]$ suffers periodic explosions, which the references call *Knaster explosions*, in which this set suddenly explodes and covers the entire complex plane in the form of an indecomposable continuum. Otherwise, using $\delta > 0$ we cover fully the region $R_{ST} \setminus B_\delta(\phi(1))$, therefore we know fully the sets $\mathfrak{J}[g_c(z)]$ and $\mathfrak{F}[g_c(z)]$ and the proof of the proposition is complete. □

4.3 General topological map of the infinite exponential

A general map which describes the local behavior of the sequence α_n on the complex plane for the corresponding sets which will be created using a given base of the exponential $\lambda = \log(c)$, for $c \in \mathbb{C}$ can be seen in [93] and [66] and is given here in Figure 3.

The colored regions are areas of period $p \geq 1$. The black pixels are points which have escaped the size bound that has been set when imaging with the program. Consequently the accuracy on these regions depends on the aforementioned escape bound set when iterating the sequence of Euler.

The neutral points of $\mathfrak{J}[g]$ are precisely the points $\phi(t)$, with t an n-th root of unity and $\phi(z) = \exp(z \cdot \exp(-z)))$, which lie in ∂R_{ST}.

The simplest case therefore is given by $p = 1$ and $c = e^{e^{-1}}$, which is a plain bouquet, with one conspicuous neutral point at the vertex of the bouquet at $h(0, c)$. The immediately next case is given by $p = 2$ and $c = e^{-e}$, which is two bouquets interacting at the neutral point $h(0, c)$. The general case for $c = \phi(t)$, with $t^p = 1$, are two bouquets which interact with period p at the neutral point $h(0, c)$. The rest of the cases for $|t| > 1$ give in general two bouquets which interact with period p at $h(0, c)$ in combination with the p-periodic attractors A_p, which are vertices of a deformed p-polygon on the complex plane.

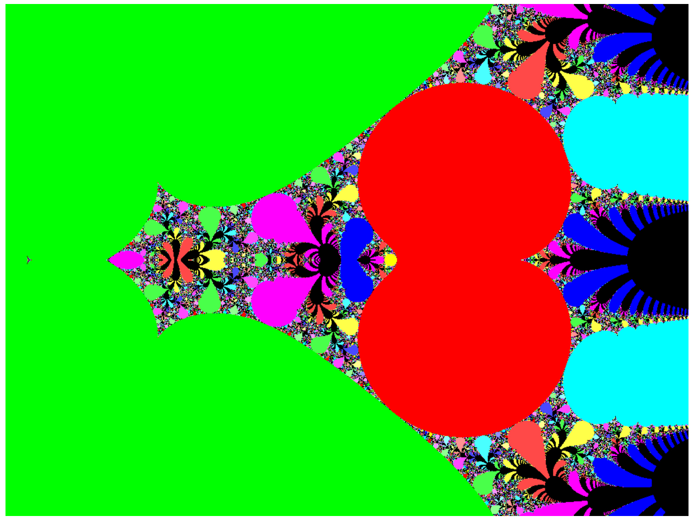

Figure 3: Parameter map of $g_c(z)$, $c \in \mathbb{C}$

When a point moves from the interior of the region R_{ST} to the exterior, during its passing from ∂R_{ST}, we observe a periodic explosion or a fork of Cantor bouquets, in which gradually show up p periodic attractors around the point $h(0,c)$, where p is the pre-period of the multiplier of c, as in [76] and [78]. In other words, we observe a Knaster explosion of period p.

The regions of the complex plane where α_n causes forks of period p is precisely the Fatou-like subset $\mathfrak{F}[g_c(z)]$ of the map of Figure 3. This subset consists of infinitely thin threads which divide the complex plane into regions where α_n is periodic. These threads create Cantor bouquets and their accuracy depends on the accuracy of the calculating program. Had the accuracy of the calculations been unlimited, the threads of the bouquets would not show at all, since the thickness of any single thread is actually zero.

Consequently, Lemma (3.3) generalizes the central Lemma (2.18) and classifies normally the periods of the sequences for any point on the complex plane with sole exception a neighborhood of $\phi(1)$. This answers Proposition (1.2) of Euler in its general form, that is for general $x = c \neq e^{e^{-1}}$ on the complex plane.

The characteristic points therefore of the topology of the infinite exponential as it follows from the above are exactly $p+1$ points: Complex infinity, towards which move all Cantor bouquet points and p attractors of the Fatou lake beds (when they exist) as they are given by Lemma 3.3, towards which move all the points of these lake beds, in spiral manner.

The torsion angle of the spiral convergence of the periodic attractors of the lake beds is a function of the neighborhood of the point of the base of the exponential map in relation to the phase angle of the boundary of the Shell-Thron region. This means that to get spiral convergence with greater or smaller torsion angle, it suffices to change a bit the angle of the pre-period of the base of the exponential map in a small neighborhood of the base, relative to the previous phase angle of the base.

4.4 Topology

Proposition (4.2) shows that the topology of the set which results from the repeated application of the exponential map, when it is not subject to Knaster explosions, is in general a p symmetry or in other words, a Fatou flower or differently, a deformed p-polygon on the complex plane or otherwise a fork of Cantor bouquets of period p.

The topology of a Cantor bouquet is given in the references above and is homeomorphic to the topology of a union of half open intervals $[\alpha, +\infty)$, with the α of the bouquet vertices in correspondence to the points of a sheared Cantor-like set $\mathfrak{CS}$ (see section 4.5).

If the bouquet is $\mathfrak{CB}_{h(k,c)} = \left\{ \bigcup\limits_k [\alpha_k, +\infty) \right\}$, in the general case, the topology of the infinite exponential, will be $\mathfrak{J}[g_c(z)] \bigcup \mathfrak{F}[g_c(z)] = \left\{ \mathfrak{CB}_{h(k,c)} \times \diamond_p \right\}$. In other words, a fork of Cantor bouquets, of period p.

Examples are given in Figures 5 and 8, 9, 10, 11, 12, 13 of the Appendix.

4.5 Fractal nature in the infinite exponential

The Julia sets here, that is the Cantor bouquets of the infinite exponential display an intense fractal character. Specifically, for given screen resolution j, which is a function of the bail-out value in the example of the code in the Appendix, Devaney represents them using "fingers" δ^j_n, $n \in \mathbb{N}$.

We note a periodic break-up of each finger δ^j into a subsequence of fingers δ^{j+1}_n, $n \in \mathbb{N}$, for each apparent finger δ^j, which repeats ad infinitum for each new finger. Consequently the following relation gives a partial cover of the threads of a bouquet at resolution level j, as $\delta^j = \bigcup_{n=0}^{\infty} \delta^{j+1}_n$. An example is given in Figure 4.

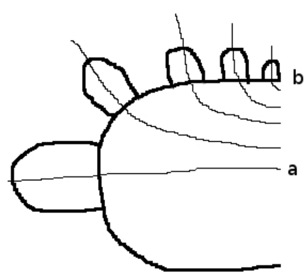

Figure 4: Devaney "finger" δ^j of Cantor bouquet

In an orthocentrically oriented finger δ^j, like that of figure 4 (oriented in this case with infinity down), the vertical width of the bouquet is bounded (see again [76], [75] and [77]) and the break up of half the finger can be set to a homeomorphic correspondence with a suitable Cantor set, on the closed vertical interval $[a, b]$, with a the vertical projection of the vertex of δ^j on the real axis and b half the horizontal dimension of the finger.

The central lines in Figure 4 show approximately the the homeomorphic correspondence implied for finger δ^j.

The dimension on either axis is additive, so if a_n and b_n, $n \in \mathbb{N}$ are the traces of the bases of the (sub-)fingers δ^{j+1} on the interval $[a, b]$, we write the recursive expression that describes the break up of finger δ^j into fingers δ^{j+1} as,

$$\dim_{H,x}(\delta^j) \sim \sum_{j=0}^{\infty} \dim_H(\delta^{j+1})_x = \sum_{n=0}^{\infty} |b_n - a_n| \tag{13}$$

The total thickness of the finger is finite $(2\,|b-a|)$ in the horizontal dimension, therefore the sequence $\dim_{H,x}(\delta^n) \sim \sum_{j=0}^{n} |b_n - a_n|$ is geometric with ratio,

$$r = \left| \frac{\dim_{H,x}(\delta^{n+1})}{\dim_{H,x}(\delta^n)} \right| = \frac{\left|\delta^{n+1}\right|_x}{\left|\delta^n\right|_x} \tag{14}$$

Note that r is independent of n (being the fixed ratio of a geometric), therefore equation (14) must be satisfied also for $n = 1$, which reveals that in the given resolution of the finger δ^j we have in our screen it suffices to measure the difference in the diameters of any two successive relative to resolution level fingers in the image of the bouquet.

If we had then at our disposal a “straight” bouquet against at least one dimension x or y, then it would suffices to measure the difference in diameter between any two such fingers in screen pixels and we would immediately get the ratio of the geometric that is the break up ratio of the Cantor bouquet by at least one direction.

There exists one and *only one* such bouquet. The bouquet produced by the exponential map with base $\lambda = e^{-1}$, which sees at point $\exp(1)$ (preperiod $p = 1$, center), which includes the interval $[e, +\infty)$. For any other base apart from this lambda the behavior of the exponential for a base in the right or left neighborhood of this point (λ) in the direction θ is unpredictable with Knaster explosions which deform the bouquets and force them to suddenly cover the entire complex plane. Such a measuring of difference between fingers therefore is not possible in images of bouquets for base $|\lambda| > 1/e$.

Yet we also have the images from all the other values of the base we calculated. No bouquet however allows accurate measuring by specific dimension, because the bouquet threads may suddenly turn unexpectedly towards a direction other than the expected one we measure in, therefore any such measuring in this direction will necessarily produce errors.

This way, the only bouquet available is that of $\lambda = 1/e$, since it’s the only one among the one’s we’ve described, for which we know that exactly one of its threads is a straight line. An image of this bouquet is given in Figure 5. The image of this bouquet however extends all the way to positive infinity (case of angle $\theta = 0$), therefore it distorts the ratio r of the geometric at least by direction θ.

The actual ratio of the geometric therefore can be measured in direction $x \geq e$ with the help of the map $\mu(z) = 1/z$, finding the inverse image relative to infinity as a fractal set using some Maple code. It is the image which shows in Figure 6.

μ is meromorphic in any neighborhood of infinity $|z| \geq R > 0$ and the exponential with base $\lambda = e^{-1}$ is also meromorphic on the interval $[e, +\infty)$, therefore the image of Figure 6 is conformal on $[\mu(R), \mu(e)]$.

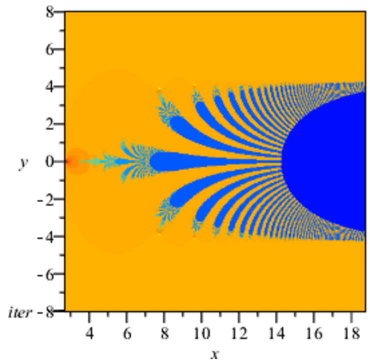

Figure 5: Cantor bouquet at $c = e$ (of base $\lambda = e^{-1}$)

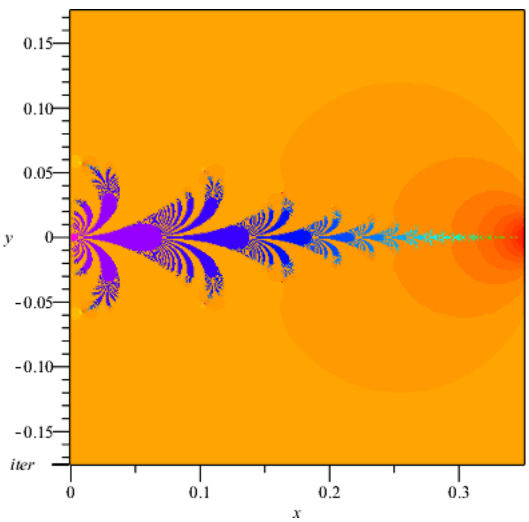

Figure 6: Cantor bouquet at $c = e$ under the map $\mu(z) = 1/z$

Therefore on this interval the dimension is given by the topology of the measure of the absolute value of a geometric with successive finger intervals $a_{n+1} = b_n$, for each $n \in \mathbb{N}$, as

$$
\begin{aligned}
|\mu(e)-\mu(R)|_x &= \sum_{n=0}^{\infty} \dim_{H,x}(\delta^n) \\
&= \sum_{n=0}^{\infty} |\delta^n|_x \\
&= \sum_{n=0}^{\infty} |b_n - a_n| \\
&= \sum_{n=0}^{\infty} |b_0 - a_0| \cdot r^n = |b_0 - a_0| \cdot \frac{1}{1-r}
\end{aligned}
\tag{15}
$$

Considering now a sequence $R_n \to \infty$, the measure of the geometric can be measured experimentally from Figure 6, either as,

$$
r_x = \lim_{R_n \to \infty} \left(1 - \frac{|\delta^0|_x}{|\mu(e) - \mu(R_n)|} \right) < 1 \tag{16}
$$

or as,

$$
r_x = \frac{\dim_{H,x}(\delta^2)}{\dim_{H,x}(\delta^1)} = \frac{|\delta^2|_x}{|\delta^1|_x} < 1 \tag{17}
$$

Using equation (17) in[2] Figure 6, we find,

$$
r_x \sim \left| \frac{490-337}{337-150} \right| = 0.8181 \tag{18}
$$

In the horizontal direction we therefore have $\dim_{H,x}(\delta) \sim r_x$. In the vertical direction (y) the dimension of the bouquet is at least that of the continuum ($[e,+\infty)$) therefore,

$$
\begin{aligned}
&1 \le \dim_{H,y}(\delta) \\
&0 \le \dim_{H,x}(\delta) \le r_x \Rightarrow \\
&1 \le \dim_{H}(\delta) \le 1 + r_x = 1.818
\end{aligned}
$$

We finally get $1 \le \dim_H(\mathfrak{CB}_e) \le 1 + r_x = 1.8181$, which means[3] that the dimension of the Cantor bouquet at e ($\lambda = e^{-1}$) is fractional with geometric ratio of decomposition $\sim r$ by at least one arbitrary direction.

Against the vertical direction then, the intersection of the bouquet $\lambda = 1/e$ is homeomorphic to a generalized[4] Cantor set, of dimension r, which is a fractal. The dimension of such sets is analyzed partly in [60] and is given as a function of a general parameter $0 < \gamma < 1$, as

[2] For the measuring a figure of greater resolution was used.

[3] The java program on page "http://www.stevec.org/fracdim/" on the net gives $r \sim 1.85$ using the box counting topology.

[4] Generalized but *symmetric* Cantor set, as the bouquet is symmetric relative to the axis $[e,+\infty)$.

$$r = \frac{\ln(2)}{\ln(\frac{1-\gamma}{2})} \tag{19}$$

In our case we have then $r \sim 0.8181$, therefore if we solve equation (17) for γ, we get $\gamma \sim 1/7$, which means that the corresponding sheared set is homeomorphic to a Cantor set which has a geometric break down ratio 1/7. The common known Cantor set has a break down ratio equal to 1/3. We can therefore say that the bouquet at e for $\lambda = 1/e$ is a fractal. It is dense by vertical section and the set $\mathfrak{CB}_e \bigcup \{+\infty\}$ is connected. if we remove however the point $\{+\infty\}$, then the set is totally disconnected.

4.6 The primitive of a Cantor bouquet

The primitive of a Cantor bouquet therefore may be revealed if we examine the image of the n-th roots of the neighborhood $B_\delta(\phi(1))$, under the map $g_c(z)$ and is presented here in Figure 7 for $n = 100$, of the neighborhood $\delta = 0.01$, of $\phi(1) = e^{e^{-1}}$. The Figure validates the aforementioned references which characterize it as homeomorphic to a C^∞ brush.

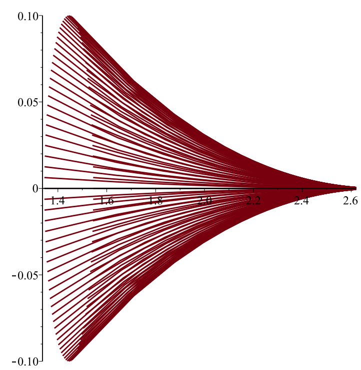

Figure 7: Primitive of Cantor bouquet at $c = e^{1/e}$

We conclude immediately that in as far as the bouquet for $\lambda = 1/e$ is concerned, all the threads of this bouquet are C^∞ curves on the complex plane.

The form of the primitive in combination to the measure of the fractal dimension, reveals fully the structure of the bouquet on the microscope. What is visible elementarily is in general a smooth (infinitely differentiable) curve-thread, which periodically explodes into a continuum of other infinitely smooth curves-threads.

The noteworthy characteristic with the threads of any bouquet then, is that only the coordinates of the thread $[e, +\infty)$ of the bouquet with $\lambda = 1/e$ are known. Any other thread is (computationally) unknown, meaning that *no point* (of any other thread) can be determined exactly[5], except the points towards which the bouquets "point" when we

[5]Except as a function of the resolution level of the screen j or equivalently of the bail-out value in the code of the Appendix

have a bouquet fork of period $p \geq 1$. In contrast with the previous characteristic, the attractors and the repellers of the Fatou lake beds can always be expressed analytically.

4.7 Pinball with the infinite exponential

We also observe that the whole topology of the infinite exponential is additionally periodic modulo $0 < \theta \leq 2\pi$ on the complex plane, as all figures display bouquets which protrude against the center under a specific angle θ. This angle is characteristic of the angle of the pre-period of the base λ of the exponential.

The previous periodicity allows us to concentrate on the topology of the Euler sequence on the complex plane modulo one turn θ which is finally equivalent to the dynamics of the orbits of a ball in a pinball game on the strip $\{z\colon |\Re(z)| \leq b_\theta\} \times \{-\infty\}$, equipped with the topology of the Fatou lake beds and highlands or Julia crevices with metric the speed of convergence/attraction (also see [87]) or repulsion to infinity and with the attractor at infinity being equivalent to gravity on a pinball table of angle θ.

Whatever is close to a Cantor bouquet sooner or later goes to complex infinity (side direction of gravity under plane angle θ relative to horizontal plane), while whatever is inside a lake bed finally falls inside a small hole like in a regular pinball machine of inclination θ, with p local holes or dingy surface ornaments.

With the exception of a neighborhood of the point $\lambda = \log(c) = e^{-1}$, for rational multiples of the pre-period angle $2\pi/p$ of the base λ of the exponential map $g_c(z) = c^z$, the topology of the infinite exponential is (at least) a Cantor bouquet at a p-furcation in a Fatou lake bed with center α, with α and $p \in \mathbb{N}$ determined analytically.

5 An indexing scheme for the HW maps

5.1 An algebraic scheme

For the complex maps log and W, their indexing scheme is the simplest possible, that is $\log(k, z)$ kai $\mathrm{W}(k, z)$, $k \in \mathbb{Z}$. There exists an indexing scheme which indexes identically the mappings HW but it is not integral. Dubinov in [204] solves Kepler's equation, using the following algebraic inversion:

$$
\begin{aligned}
E - \epsilon \cdot \sin(E) &= M \Rightarrow \\
E\left(1 - \epsilon \frac{\sin(E)}{E}\right) &= M \Rightarrow \\
E \cdot e^{\log 1 - \epsilon \cdot \operatorname{sinc} E} &= M \Rightarrow \\
&E = \mathrm{HW}\left[\log(1 - \epsilon \cdot \operatorname{sinc}(x)); M\right]
\end{aligned}
\tag{20}
$$

The inversion above can be generalized producing a removable pole at z_0 of multiplicity n. Setting $w = (z - z_0)^n$, with z_0 such that $f(z_0) = y$, we have:

$$
\begin{aligned}
f(z) &= y \Rightarrow \\
(z-z_0)^n \cdot \frac{f(z)}{(z-z_0)^n} &= y \Rightarrow \\
w \cdot e^{\log\left(\frac{f(z)}{w}\right)} &= y \Rightarrow \\
w &= \mathrm{HW}\left[\log\left(\frac{f(z)}{w}\right); y\right] \Rightarrow \\
(z-z_0)^n &= \mathrm{HW}\left[\log\left(\frac{f(z)}{(z-z_0)^n}\right); y\right] \Rightarrow \\
z &= \mathrm{HW}\left[\log\left(\frac{f(z)}{(z-z_0)^n}\right); y\right]^{\frac{1}{n}} + z_0
\end{aligned}
\tag{21}
$$

The scheme above gives an index into the set of the HW functions, in the form of a functional parameter as $\log\left(\frac{f(z)}{(z-z_0)^n}\right)$. Now, if we know $f(z)$, this scheme can give identities which must hold identifying this way the corresponding function.

We can now list the most important categories of complex functions based on this index.

5.2 Polynomial functions

Suppose then that $f(z) = \prod_{k=1}^{N}(z-z_k)^{n_k}$. Keeping k fixed and setting $w = (z-z_k)^{n_k}$, we have,

$$
\begin{aligned}
w &= \mathrm{HW}\left[\log\left(\frac{f(z)}{w}\right); y\right] \Rightarrow \\
z &= \mathrm{HW}\left[\log\left(\frac{f(z)}{(z-z_k)^{n_k}}\right); y\right]^{\frac{1}{n_k}} + z_k \Rightarrow \\
z &= \mathrm{HW}\left[\log(f(z)) - n_k \log(z-z_k); y\right]^{\frac{1}{n_k}} + z_k
\end{aligned}
\tag{22}
$$

Theorem 5.1 *If $f(z) = \prod_{k=1}^{N}(z-z_k)^{n_k}$ is a complex polynomial function, then the inverse of $f(z)$ relative to y is given by the function* HW *and the last equation of* (22)*, whose Riemann surface has at most $m = \sum_{k=1}^{N} n_k$ branches, indexed by m, with $k \in \mathbb{N}$.*

Proof: The last expression of (22) is true for any $k \in \{1, 2, \ldots, N\}$, therefore the multiplicity is at least m because for each k the multiplicity is at least n_k and each n_k may give different branches. This means that the expression can index fully all the branches of the corresponding HW using only an integral index k. □

This means,

Theorem 5.2 *If $f(z)$ is a complex polynomial function, the roots of $f(z) = y$ are given directly by a suitable* HW *function.*

Proof: Using equation (9) of Definition (3.2), follows that for each HW, $\text{HW}(\ldots; 0) = 0$, therefore calculating the corresponding HW of the last equations in (22) at $y = 0$, forces $z = z_k$ and these are the roots of $f(z) = y$. Therefore, we can extract all the roots of equation $f(z) = y$, manually. The first root, suppose z_1, is extracted as,

$$\begin{aligned} z_1 &= \text{HW}\left[\log\left(\frac{f(z)}{z}\right); y\right] \\ g_1(z) &= \frac{f(z) - y}{z - z_1} \end{aligned}$$

Having the root z_1, the rest of the roots can be extracted recursively for $1 \leq k \leq N-1$ as,

$$\begin{aligned} z_{k+1} &= \lim_{\epsilon \to 0^+} \text{HW}\left[\log\left(\frac{g_k(z)}{z}\right); \epsilon\right] \\ g_{k+1}(z) &= \frac{g_k(z)}{z - z_{k+1}} \end{aligned}$$

and the Theorem follows. □

Example: Using the program with five decimal digits accuracy,

```
y:=2;
f:=z->(z-2)*(z-3)*(z-5);
z1:=HW(log(f(z)/z),y,10);
g1:=z->(f(z)-y)/(z-z1);
z2:=HW(log(g1(z)/z),1e-20,10);
g2:=z->g1(z)/(z-z2);
z3:=HW(log(g2(z)/z),1e-20,10);
```

gives:

$$\begin{aligned} z_1 &\simeq 2.36523 - 0.69160i \\ z_2 &\simeq 2.36523 + 0.69160i \\ z_3 &\simeq 5.26953 \end{aligned}$$

Using the program for an approximate solution with Maple,

```
solve(f(z)=y,z);
evalf(%);
```

gives:
5.26953, 2.36523+0.69160i, 2.36523-0.69160i.

5.3 Rational functions

We suppose that $f(z) = P(z)/Q(z)$, with $P(z)$, $Q(z)$ polynomial functions. We have similar results here.

Theorem 5.3 *If $f(z) = P(z)/Q(z)$ is a complex rational function such that $N = \max\{\deg(P), \deg(Q)\}$, then the inverse of $f(z)$ relative to y is given by:*

$$z = \mathrm{HW}\left[\log\left(\frac{P(z) - y \cdot Q(z)}{(z - z_k)^{n_k}}\right); y\right]^{\frac{1}{n_k}} + z_k$$

whose Riemann surface has at most $m = \sum_{k=1}^{N} n_k$ branches, indexed by m, with $k \in \mathbb{N}$.

Proof: If $F(z) = P(z) - y \cdot Q(z)$, then $F(z)$ is a polynomial of degree N, in which case the Theorem follows similarly, with $f(z)$ replaced by $F(z)$. $\square$

Theorem 5.4 *If $f(z)$ is a complex rational function, the roots of $f(z) = y$ can be given by a suitable* HW *function.*

Proof: Similarly, if $F(z) = P(z) - y \cdot Q(z)$, then $F(z)$ is polyonymic of degree N, therefore we can extract its roots as:

$$z_1 = \lim_{\epsilon \to 0^+} \mathrm{HW}\left[\log\left(\frac{F(z)}{z}\right); \epsilon\right]$$
$$g_1(z) = \frac{F(z)}{z - z_1}$$

Having z_1, the rest of the roots can be extracted recursively for $1 \leq k \leq N - 1$ as,

$$z_{k+1} = \lim_{\epsilon \to 0^+} \mathrm{HW}\left[\log\left(\frac{g_k(z)}{z}\right); \epsilon\right]$$
$$g_{k+1}(z) = \frac{g_k(z)}{z - z_{k+1}}$$

and the Theorem follows. $\square$

Example: Using the program to five digits of accuracy,

```
y:=2;
f:=z->(z-2)*(z-3)/(z-5)/(z-1);
P:=unapply(numer(f(z)),z);
Q:=unapply(denom(f(z)),z);
F:=unapply(P(z)-y*Q(z),z);
z1:=HW(log(F(z)/z),1e-10,10);
g1:=z->F(z)/(z-z1);
z2:=HW(log(g1(z)/z),1e-10,10);
```

gives:

$$z_1 \simeq 6.37228$$
$$z_2 \simeq 0.62771$$

Using Maple approximation code,

```
solve(F(z)=y,z);
evalf(%);
```

gives:
6.37228, 0.62771.
We observe that when $Q(z) = 1$, this case reduces to the previous.

5.4 Analytic functions

For an analytic function $f(z) = \sum_{n=0}^{\infty} \alpha_n \cdot (z - z_0)^n$ in some region $D \subseteq \mathbb{C}$, with $z_0 \in D$, we have similar results.

Theorem 5.5 *If $f(z) = \sum_{n=0}^{\infty} \alpha_n \cdot (z - z_0)^n$ is a complex analytic function, then the inverse of $f(z)$ relative to y is given by a suitable* HW *function:*

$$z = \mathrm{HW}\left[\log\left(\frac{f(z)}{(z-z_k)^{n_k}}\right); y\right]^{\frac{1}{n_k}} + z_k$$

whose Riemann surface has infinitely many branches given by $n \in \mathbb{N}$.

Proof: Suppose $T_N(z) = \sum_{n=0}^{N} \alpha_n \cdot (z - z_0)^n$, is the corresponding Taylor polynomial of degree N. Then $T_N(z)$ is obviously a polynomial function, therefore the inverse of $T_N(z)$ relative to y is given again by Theorem (5.1).

$$z = \mathrm{HW}\left[\log\left(\frac{T_N(z)}{(z-z_k)^{n_k}}\right); y\right]^{\frac{1}{n_k}} + z_k \tag{23}$$

$T_N(z) \to f(z)$ uniformly in compact subsets and the HW are analytic ([208]), therefore (23) implies that the inverse is given by:

$$\begin{aligned} z &= \lim_{N\to\infty} \mathrm{HW}\left[\log\left(\frac{T_N(z)}{(z-z_k)^{n_k}}\right); y\right]^{\frac{1}{n_k}} + z_k \Rightarrow \\ z &= \mathrm{HW}\left[\log\left(\frac{\lim_{N\to\infty} T_N(z)}{(z-z_k)^{n_k}}\right); y\right]^{\frac{1}{n_k}} + z_k \Rightarrow \\ z &= \mathrm{HW}\left[\log\left(\frac{f(z)}{(z-z_k)^{n_k}}\right); y\right]^{\frac{1}{n_k}} + z_k \end{aligned} \tag{24}$$

and the Theorem follows. $\square$

We observe that in this case the inverse function has infinitely many branches, since N is not bounded.

Theorem 5.6 *If $f(z)$ is a complex analytic function, the roots of $f(z) = y$ are given again by a* HW *function.*

Proof: We can extract the roots as:

$$\begin{aligned} z_1 &= \mathrm{HW}\left[\log\left(\frac{f(z)}{z}\right); y\right] \\ g_1(z) &= \frac{f(z) - y}{z - z_1} \end{aligned} \tag{25}$$

The rest of the roots can be again extracted recursively for $1 \leq k$ as,

$$\begin{aligned} z_{k+1} &= \lim_{\epsilon \to 0^+} \mathrm{HW}\left[\log\left(\frac{g_k(z)}{z}\right); \epsilon\right] \\ g_{k+1}(z) &= \frac{g_k(z)}{z - z_{k+1}} \end{aligned} \tag{26}$$

and the Theorem follows. $\square$

Example: Using the program to five decimals of accuracy,

```
y:=1/2;
f:=z->sin(z);
z1:=HW(log(f(z)/z),y,10);
g1:=z->(f(z)-y)/(z-z1);
z2:=HW(log(g1(z)/z),1e-20,10);
g2:=z->g1(z)/(z-z2);
z3:=HW(log(g2(z)/z),1e-20,10);
```

gives:

$$\begin{aligned} z_1 &\simeq 0.52359 \\ z_2 &\simeq 2.61799 \\ z_3 &\simeq -3.66519 \\ z_4 &\simeq \cdots \end{aligned}$$

The results are approximations of the numbers $\pi/6$, $5\pi/6$, $-7\pi/6$,..., which are the roots of $\sin(z) = 1/2$.

5.5 HW **functional index**

An open problem set in [207, 1114-1115] is whether there is a way to effectively index the numbering of the branches of the HW functions. With the following Theorem we show that the answer is affirmative.

Theorem 5.7 *If $f(z)$ is a complex function and $z_k \in \mathbb{C}$, $k \in \mathbb{N}$, such that $f(z_k) = y$ and suppose $g_k(z)$ follows as in equations* (25) - (26)*. Then, if* HW *is the inverse of $f(z)$ relative to y, the following scheme covers all the branches of this inverse of $f(z)$:*

$$z_{k+1} = \begin{cases} \mathrm{HW}\left[(0,)\log\left(\frac{f(z)}{z}\right);y\right], \ \textit{an}\ k=0, \\ \lim\limits_{\epsilon\to 0^+} \mathrm{HW}\left[k, \log\left(\frac{g_k(z)}{z}\right);\epsilon\right], \ \textit{an}\ k>0. \end{cases}$$

Proof: The scheme is validated in the previous sections. For a spcific analytic f expanded around z_k, we define $F(z) = \log\left(\frac{f(z)}{(z-z_k)^{n_k}}\right)$. The map F creates a Laurent series with residue $\exp(a_{n_k})$, which is gotten from the HW through the Residue Theorem of Cauchy for f, with winding number a_{n_k} around z_k. Consequently, the repeated application of F (via g_k) for $z = z_k$, extracts recursively all the roots z_k of the inverse and as such it can be used as an index for the corresponding Riemann surface. □

The above description of the inverse calculates correctly branches with multiplicity greater than 1, when the corresponding roots have this multiplicity.

Example 2: Using the code with five decimal accuracy,

```
y:=2;
f:=z->z^3-4*z^2+5*z;
z1:=HW(log(f(z)/z),y,10);
g1:=z->(f(z)-y)/(z-z1);
z2:=HW(log(f1(z)/z),1e-20,10);
g2:=z->g1(z)/(z-z2);
z3:=HW(log(g2(z)/z),1e-20,10);
```

gives:

$$\begin{aligned} z_1 &\simeq 2 \\ z_2 &\simeq 1.00005 \\ z_3 &\simeq 1.00000 \end{aligned}$$

In the previous case it is $f(z) - y = (z-1)^2(z-2)$, therefore the multiplicity of the root 1 is indeed 2.

6 The generalized infinite exponential

6.1 Definitions

The general infinite exponential is analyzed in a limited way in [209] ([18],[178], [179]) and in [122]. It is defined recursively:

Definition 6.1 *If $Z_k = \{z_1, z_2, \ldots, z_k\}$, $k \in \mathbb{N}$, with $z_k \in \mathbb{C} \setminus \{x \in \mathbb{R} \colon x \leq 0\}$ and $n \leq |Z_k| = k$,*

$$e_n(Z_k) = \begin{cases} z_k & , \text{ if } n = 1, \\ z_{k-n+1}^{e_{n-1}(Z_k)} & , \text{ if } n > 1. \end{cases}$$

Whenever we need to study convergence of the previous exponential, we simply examine the values of the sequence $b_n = e_n(Z_n)$. Unwinding recursively b_n, $n \in \mathbb{N}$, using Definition (6.1), produces the terms: $z_1, z_1^{z_2}, z_1^{z_2^{z_3}}, \ldots, z_1^{z_2^{z_3^{\cdot^{\cdot^{\cdot^{z_n}}}}}}$.

There is however a dual definition which reverses the indexes, as

Definition 6.2 *Given a list $Z_k = \{z_1, z_2, \ldots, z_k\}$, $k \in \mathbb{N}$, with $z_k \in \mathbb{C} \setminus \{x \in \mathbb{R} \colon x \leq 0\}$ and $n \leq |Z_k| = k$,*

$$e_n^*(Z_k) = \begin{cases} z_1 & , \text{ if } n = 1, \\ z_n^{e_{n-1}^*(Z_k)} & , \text{ if } n > 1. \end{cases}$$

Analyzing recursively the sequence $b_n^* = e_n^*(Z_n)$, $n \in \mathbb{N}$, using Definition (6.2), produces the sequence $z_1, z_2^{z_1}, z_3^{z_2^{z_1}}, \ldots, z_n^{z_{n-1}^{z_{n-2}^{\cdot^{\cdot^{\cdot^{z_1}}}}}}$.

Whenever the general infinite exponential converges, we write,

Definition 6.3 *Whenever the following limit exists finitely,*

$$e_\infty(Z_\infty) = \lim_{n\to\infty} e_n(Z_n) = \lim_{n\to\infty} b_n = b_\infty$$
$$e_\infty^*(Z_\infty) = \lim_{n\to\infty} e_n^*(Z_n) = \lim_{n\to\infty} b_n^* = b_\infty^*$$

The references ([18]) show the following (in relation to Definitions (6.1) and (6.2)):

Theorem 6.4 ***(Barrow)*** *The sequence b_n converges, if and only if:*

- $\exists n_0 \colon \forall n \geq n_0 \colon b_n \in [(1/e)^e, e^{(1/e)}]$.
- *z_n converges.*

Theorem (6.4) can be now generalized, since we now know the full extent of the R_{ST} region:

Theorem 6.5 ***(Barrow)*** *Sequence b_n converges, if and only if:*

- $\exists n_0 \colon \forall n \geq n_0 \colon b_n \in R_{ST}$.
- *z_n converges.*

We are interested in finding interesting expressions for the limits b_∞ and b_∞^*.

6.2 Ascending indexes

For the first case (Definition (6.1)), we set $G_n(z) = e^*_{n+1}(L_n \cup \{z\})$, in which case we get the iteration:

$$
\begin{aligned}
G_n(z) &= y \Rightarrow \\
\log(G_n(z)) &= \log(y) \Rightarrow \\
G_{n-1}(z) &= \frac{\log(y)}{\log(z_n)} \Rightarrow \\
\log(G_{n-1}(z)) &= \log\left(\frac{\log(y)}{\log(z_n)}\right) \Rightarrow \\
G_{n-2}(z) &= \frac{\log(G_{n-1}(z))}{\log(z_{n-1})} \Rightarrow \\
&\dots \\
G_{n-k+1}(z) &= \frac{\log(G_{n-k}(z))}{\log(z_{n-k})} \\
G_{m+1}(z) &= \frac{\log(G_m(z))}{\log(z_m)}
\end{aligned}
\tag{27}
$$

Suppose $z_0 = \lim_{n\to\infty} z_n$. We select k such that $m = n - k \geq n_0$. We can take limits for $m \to \infty$ in the last of (27) and solve for W:

$$
\begin{aligned}
G(z) &= \frac{\log(G(z))}{\log(z_0)} \Rightarrow \\
G(z) &= \frac{\mathrm{W}(-\log(z_0))}{-\log(z_0)} = h(z_0)
\end{aligned}
\tag{28}
$$

Equations (28) show,

Theorem 6.6 *If $z_0 = \lim_{n\to\infty} z_n$, then for $\epsilon > 0$, there exists n_0, such that for $n \geq n_0$, $|b^*_n - h(z_0)| < \epsilon$.*

Proof: For any $\epsilon > 0$, we can find n_0 sufficiently big that guarantees $|z_n - z_0| < \epsilon$ and then the bases of the tower until z_{n_0} can be expressed as $z_0 + \epsilon$. The elements further than z_{n_0} can be gradually sent to infinity, by choosing big n_0 and then the limit of the tower will approach the number $h(z_0 + \epsilon)$. The Theorem follows by forcing $\epsilon \to 0$. $\square$

6.3 Descending indexes

For the dual case (Definition (6.2)), if $Z_k = \{z_1, z_2, \dots, z_k\}$, transforming as $w_m = z_{n-m+1}$ and as $Z^*_k = \{w_1, w_2, \dots, w_k\}$ to change the order of the indexes, we set $G_n(z) = e^*_n(Z^*_n \cup \{z\})$, in which case we get the iteration:

$$
\begin{aligned}
G_n(z) &= y \Rightarrow \\
\log(G_n(z)) &= \log(y) \Rightarrow \\
G_{n-1}(z) &= \frac{\log(y)}{\log(w_n)} \Rightarrow \\
\log(G_{n-1}(z)) &= \log\left(\frac{\log(y)}{\log(w_n)}\right) \Rightarrow \\
G_{n-2}(z) &= \frac{\log(G_{n-1}(z))}{\log(w_{n-1})} \Rightarrow \\
&\dots \\
G_{n-k+1}(z) &= \frac{\log(G_{n-k}(z))}{\log(w_{n-k})} \\
G_{m+1}(z) &= \frac{\log(G_m(z))}{\log(w_m)}
\end{aligned}
\tag{29}
$$

Suppose $z_0 = \lim_{n\to\infty} z_n$. We choose k such that $m = n - k \geq n_0$. Similarly, we can get the limit for $m \to \infty$ in the last of (29) and to solve for W:

$$
\begin{aligned}
G(z) &= \frac{\log(G(z))}{\log(z_0)} \Rightarrow \\
G(z) &= \frac{\mathrm{W}(-\log(z_0))}{-\log(z_0)} = h(z_0)
\end{aligned}
\tag{30}
$$

Equations (30) show,

Theorem 6.7 *If $z_0 = \lim_{n\to\infty} z_n$, for each $\epsilon > 0$, there exists n_0, such that for each $n \geq n_0$, $|b_n - e_{n_0+1}(Z_{n_0} \cup \{h(z_0)\})| < \epsilon$.*

Proof: Similarly, given $\epsilon > 0$, we can find n_0 big enough that guarantees $|z_n - z_0| < \epsilon$, and then all the terms higher than z_{n_0} can be approximated as $h(z_0+\epsilon)$. The tower below z_{n_0}, leaves the elements untouched, therefore the approximation will be $z_1^{z_2^{\cdot^{\cdot^{\cdot^{z_{n_0}^{h(z_0)+\epsilon}}}}}}$. The Theorem follows by forcing $\epsilon \to 0$. $\square$

Theorems (6.6) and (6.7) in combination with Proposition (4.2) answer Proposition (1.2) of Euler in its general form. Specifically,

Corollary 6.8 *Suppose the sequence γ_n, $n \in \mathbb{N}$ with $\gamma_n \to c \in \mathbb{C}$. The behavior of the sequence α_n with $\alpha_{n+1} = \gamma_n^{\alpha_n}$, $n \in \mathbb{N}$, is described by the sets $\mathfrak{J}[g_{c+\epsilon}]$ and $\mathfrak{F}[g_{c+\epsilon}]$, with $\epsilon > 0$ from Theorem (6.6).*

7 Appendix: programming/results

Executable Maple code is attached in a separate .pdf document, to which the author refers for the validation of the Corollaries and the creation of the Figures of the dissertation. In the figures, the first column contains an attractor $A_1 = h(0, c)$, the second column contains the neutral point $N_1 = h(0, c)$ and the third column contains the repeller $R_1 = h(0, c)$ and the periodic attractors A_p of Proposition (4.2), distributed around the repeller R_1.

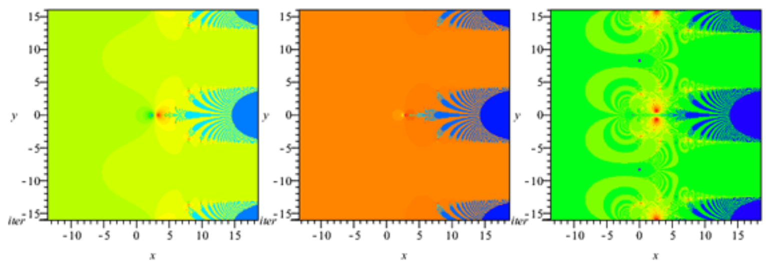

Figure 8: Preperiod $p = 1$, with $\lambda = \log(c) = t/e^t$, $|t| < 1$, $t = 1$ and $|\lambda| > 1/e$

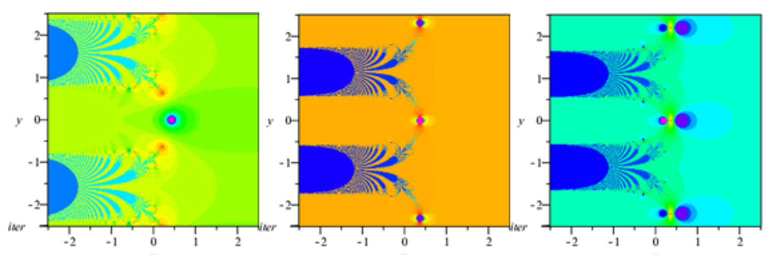

Figure 9: Preperiod $p = 2$, with $\lambda = \log(c) = t/e^t$, $|t| < 1$, $t = e^{\pi i} = -1$ and $|t| > 1$

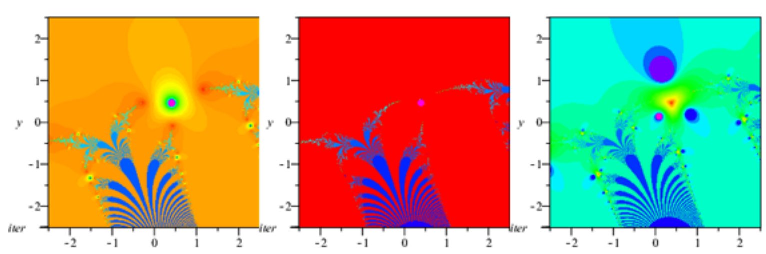

Figure 10: Preperiod $p = 3$, with $\lambda = \log(c) = t/e^t$, $|t| < 1$, $t = e^{2\pi/3i}$ and $|t| > 1$

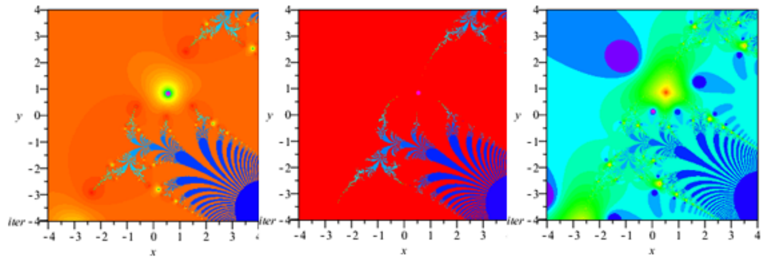

Figure 11: Preperiod $p = 4$, with $\lambda = \log(c) = t/e^t$, $|t| < 1$, $t = e^{\pi/2i}$ and $|t| > 1$

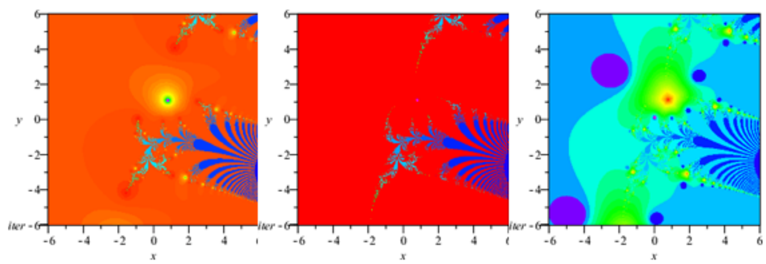

Figure 12: Preperiod $p = 5$, with $\lambda = \log(c) = t/e^t$, $|t| < 1$, $t = e^{2\pi/5i}$ and $|t| > 1$

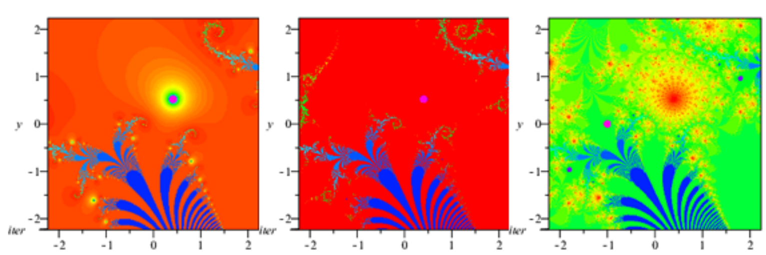

Figure 13: Preperiod $p = \pi$, with $\lambda = \log(c) = t/e^t$, $|t| < 1$, $t = e^{2i}$ and $|t| > 1$

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9 Bibliography

References

[1] W.B. Manhard 2d. Is exponentiation commutative? *Mathematics Teacher*, 74(1):56–60, Jan. 1981.

[2] 30 September 1905 8-th regular meeting of the San Franscisco Section of the AMS, University of California. Comment 8. *Bulletin of the American Mathematical Society*, 12:115, 1905-1906.

[3] D.S. Alexander. *A History of Complex Dynamics, From Schroeder to Fatou and Julia*. Friedrich Vieweg and Sohn Verlagsgesellschaft mbH, Oct. 1994.

[4] A.O. Allen. e^π or π^e? *Journal of Recreational Mathematics*, 2(4):255–256, October 1969.

[5] M.M. Alliaume. Sure le developpment en serie de la racine d' une equation. *Mathesis Requeil Mathematique*, 44:163–166, 1930.

[6] J.C. Appleby. Notes on hexponentiation. *The Mathematical Gazette*, 79(484):x–y, 1995.

[7] R.C. Archibald and R. Clare. Problem 136. *Mathematics of Computation*, 6(39):204, July 1952.

[8] F. Arjomandi. Problem 27.1. *Mathematical Spectrum*, 27(3):68, 1994-1995.

[9] J.M. Ash. The limit of $x^{x^{\cdots^x}}$ as x tends to zero. *Mathematics Magazine*, 69:207–209, 1996.

[10] A.L. Baker. Functional exponents. *School Science and Mathematics*, 8(3):225–227, Mar. 1908.

[11] I.N. Baker and P.J. Rippon. Convergence of infinite exponentials. *Annales Academiae Scientiarum Fennicae. Series A AI Mathematics*, pages 179–186, 1983.

[12] I.N. Baker and P.J. Rippon. A note on complex iteration. *American Mathematical Monthly*, 92:501–504, 1985.

[13] I.N. Baker and P.J. Rippon. A note on infinite exponentials. *The Fibonacci Quarterly*, 23:106–112, 1985.

[14] I.N. Baker and P.J. Rippon. Iterating exponential functions with cyclic exponents. *Mathematical Proceedings of the Cambridge Philosophical Society*, 105:359–375, 1989.

[15] I.N. Baker and P.J. Rippon. Towers of exponents and other composite maps. *Complex Variables*, 12:181–200, 1989.

[16] E. Barbette. Des progressions logarithmiques. *Mathesis Requeil Mathematique*, 8(2):135–137, 1898.

[17] M. Barnsley. *Fractals Everywhere.* Academic Press, Inc., Cambridge, MA., 1988.

[18] D.F. Barrow. Infinite exponentials. *American Mathematical Monthly*, 43:150–160, 1936.

[19] K.-H. Becker and M. Dörfler. *Dynamical Systems and Fractals.* Cambridge University Press, 1990.

[20] S. Bedding and K. Briggs. Iteration of quaternion functions. *American Mathematical Monthly*, 8(103):654–664, Oct. 1996.

[21] E.T. Bell. Iterated exponential numbers. *Bulletin of the American Mathematical Society*, 43:774–775, 1937.

[22] E.T. Bell. The iterated exponential integers. *The Annals of Mathematics, Second Series*, 39:539–557, 1938.

[23] W.W. Beman. An infinite exponential. *Mathematical Spectrum*, 27(1):22, 1994-1995.

[24] W.W. Beman. Problem 389. *American Mathematical Monthly*, 21(1):23, January 1914.

[25] C.M. Bender. *Advanced Mathematical Methods for Scientists and Engineers, Approximate Solution of Nonlinear Differential Equations.* McGraw Hill, 1978.

[26] A.A. Bennet. The iteration of functions of one variable. *The Annals of Mathematics, Second Series*, 17:23–60, 1915.

[27] A.A. Bennet. Note on an operation of the third grade. *The Annals of Mathematics, Second Series*, 17:74–75, 1915.

[28] A. Bennett. Note on an operation of the third grade. *The Annals of Mathematics, Second Series*, 17(2):74–75, Dec. 1915.

[29] A. Bennett. The iteration of functions of one variable. *The Annals of Mathematics, Second Series*, 17(1):23–60, Sep. 1915.

[30] W. Bergweiler. Periodic points of entire functions. *Complex Variables Theory Appl.*, 17:57–72, 1991.

[31] W. Bergweiler. Iteration of meromorphic functions. *Bulletin of the American Mathematical Society*, 29:151–188, 1993.

[32] B.C. Berndt. Ramanujan's Quarterly Reports. *Bull. London Math. Soc.*, 16(1984):469–489, Apr. 1983.

[33] H.J.M. Bos. Johann Bernoulli on exponential curves, ca. 1695 innovation and habituation in the transition from explicit constructions to implicit functions. *Nieuw Archief voor WisKunde*, 14(4):1–19, 1996.

[34] L. Brand. Binomial expansions in factorial powers. *American Mathematical Monthly*, 67(10):953–957, Dec. 1960.

[35] L. Brand. The roots of a quaternion. *The American Mathematical Monthly*, 49(8):519–520, Oct. 1942.

[36] B. Branner. Holomorphic dynamical systems in the complex plane an introduction, 1995. Proceedings of the 7^{th} EWM meeting, Madrid.

[37] N. Bromer. Superexponentiation. *Mathematics Magazine*, 60:169–174, 1987.

[38] H.J. Brothers and J.A. Knox. New closed-form approximations to the logarithmic constant e. *Mathematical Intelligencer*, 20(4):25–29, Fall 1998.

[39] M. Brozinsky. Problem 4394. *School Science and Mathematics*, 93(6):340, Oct. 1993.

[40] B.W. Brunson. The partial order of iterated exponentials. *American Mathematical Monthly*, 93:779–786, 1986.

[41] E.P. Bugdanoff. Problem 3515 [1931,462]. *American Mathematical Monthly*, 39(9):552–555, Nov. 1932.

[42] A.P. Bulanov. Iterated cyclic exponentials and power functions with extra-periodic first coefficients. *Sb. Math*, 201(1):23:23–55, 2010.

[43] L.E. Bush. The 1961 william lowell putnam mathematical competition. *American Mathematical Monthly*, 69:759–767, 1962.

[44] F.A. Butter. A note on the equation $x^y = y^x$. *American Mathematical Monthly*, 46(6):316–317, June/July 1939.

[45] W.D. Cairns. Problem 422. *American Mathematical Monthly*, 22(4):133, Apr. 1915.

[46] F. Cajori. History of the exponential and logarithmic concepts (part 4). *American Mathematical Monthly*, 20(4):107–117, Apr. 1913.

[47] F. Cajori. History of the exponential and logarithmic concepts (part 2). *American Mathematical Monthly*, 20(2):35–37, Feb. 1913.

[48] F. Cajori. History of the exponential and logarithmic concepts. *American Mathematical Monthly*, 20(1):5–14, Jan. 1913.

[49] F. Cajori. History of the exponential and logarithmic concepts (part 6). *American Mathematical Monthly*, 20(6):173–182, June 1913.

[50] F. Cajori. Errata in the february issue. *American Mathematical Monthly*, 20(3):104, Mar. 1913.

[51] F. Cajori. History of the exponential and logarithmic concepts (part 3). *American Mathematical Monthly*, 20(3):75–84, Mar. 1913.

[52] F. Cajori. Napier's logarithmic concept. *American Mathematical Monthly*, 23(3):71–72, Mar. 1916.

[53] F. Cajori. History of the exponential and logarithmic concepts (part 5). *American Mathematical Monthly*, 20(5):148–151, May. 1913.

[54] F. Cajori. History of the exponential and logarithmic concepts (part 7). *American Mathematical Monthly*, 20(7):205–210, Sep. 1913.

[55] R.D. Carmichael. On a certain class of curves given by transcendental equations. *American Mathematical Monthly*, 13:221–226, 1906.

[56] R.D. Carmichael. On certain transcendental functions defined by a symbolic equation. *American Mathematical Monthly*, 15:78–83, 1908.

[57] R.D. Carmichael. Problem 275. *American Mathematical Monthly*, 14(2):27, Feb. 1907.

[58] G.F. Carrier, M. Krook, and C.E. Pearson. *Functions of a Complex Variable*. McGraw-Hill Book Company, New York, 1966.

[59] H.S. Carslaw. Relating to Napier's logarithmic concept. *American Mathematical Monthly*, 23(8):310–315, Oct. 1913. (Comments on the Cajori article and a response by professor Cajori).

[60] A.Yu. Cherny, E.M. Anitas, A.I. Kuklin, M. Balasoiu, and V.A. Osipov. Scattering from generalized cantor fractals. http://arxiv.org/abs/0911.2497.

[61] R.V. Churchill, J.W. Brown, and R.F. Verhey. *Complex Variables and Applications*. McGraw-Hill, New York, 1974.

[62] Jr. C.J. Everett. An exponential diophantine equation. *American Mathematical Monthly*, 43(4):229–230, Apr. 1936.

[63] P. Colwell. Bessel functions and Kepler's equation. *American Mathematical Monthly*, 99(1):45–48, Jan. 1992.

[64] S.D. Conte and Carl de Boor. *Elementary Numerical Analysis*. McGraw-Hill Book Company, New York, 1980.

[65] R. Cooper. The relative growth of some rapidly increasing sequences. *Journal of the London Mathematical Society*, 29(1):59–62, 1954.

[66] R.M. Corless. Lambert W Function, 2000. http://www.apmaths.uwo.ca/rcorless/frames/PAPERS/LambertW.

[67] R.M. Corless, G.H. Gonnet, D.E. Hare, D.J. Jeffrey, and D.E. Knuth. On the Lambert W function. *Advances in Computational Mathematics*, 5:329–359, 1996.

[68] R.M. Corless, D.J. Jeffrey, and D.E. Knuth. A sequence of series for the Lambert W function. *Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation, Maui, Hawaii. New York: ACM Press*, pages 197–204, 1997.

[69] Alex D.D. Craik. Prehistory of Faá Di Bruno's formula. *American Mathematical Monthly*, 112(2):217–234, Feb. 2005.

[70] R.E. Crandall and M.M. Colgrove. *Scientific Programming with Macintosh Pascal.* Wiley Press, 1986.

[71] C.C. Cross. Problem 93. *American Mathematical Monthly*, 6(8/9):199–201, Aug./Sep. 1899.

[72] P.J. Davis. Leohard Euler's integral: A historical profile of the Gamma function. *American Mathematical Monthly*, 66(10):849–869, Dec. 1959.

[73] C. A. Deavours. The quaternion calculus. *The American Mathematical Monthly*, 9(80):995–1008, 11 Nov., 1973.

[74] E. Delville. Nombres egaux a leurs logarithmes. *Mathesis*, 4(2):264, 1912.

[75] R.L. Devaney. Accessible points in the julia sets of stable exponentials. *Discrete and Continuous Dynamical Systems*, 1:299–318, 2001.

[76] R.L. Devaney. Cantor bouquets, explosions and knaster continua: Dynamics of complex exponentials. *Topology and its Applications*, 110:133–161, 2001.

[77] R.L. Devaney. Cantor and Sierpinski, Julia and Fatou: Complex topology meets Complex dynamics. *Notices of the American Mathematical Society*, 51(1):9–15, 2004.

[78] R.L. Devaney. *Complex Exponential Dynamics (in Handbook of Dynamical Systems), Vol. 3.* Eds. H. Broer, F. Takens, B. Hasselblatt, 2010.

[79] A.C. Dixon. On Bürmann's theorem. *Proc. London Math. Soc.*, 34:151–153, 1902.

[80] A.E. Dubinov, I.D. Dubinova, and S.K. Saikov. *Lambert W function: Table of Integrals and Other Mathematical Properties [in Russian].* Sarov Physical and Technical Institute, Sarov, 2004.

[81] R.M. Dudley. Problem 5098 [1963,445]. *American Mathematical Monthly*, 71(5):563, May. 1964.

[82] Jr. A. M. E.D.Roe. Problem 68. *American Mathematical Monthly*, 5(4):110, Apr. 1898.

[83] G. Eisenstein. Entwicklund von $a^{a^{a^{\cdots}}}$. *Journal Fur Die Reine Und Angewandte Mathematik*, 28(8):49–52, 1944.

[84] G. Enestrom. Es ware ratsam, den passus. *Bibliotheca Mathematica*, 13(3):270, 1912-1913.

[85] L. Euler. De formulis exponentialibus replicatus. *Acta Sci. Petropol.*, 1:38–60, 1778.

[86] C.L. Evans. Interesting logs, reader's reflection column. *Mathematics Teacher*, 72(7):489, Oct. 1979.

[87] L. Han F. Gao and K. Schilling. On the rate of convergence of iterated exponentials. *Journal of Applied Mathematics and Computing*, pages 1–8, Sep. 2011.

[88] M.J. Farkas. Sur le fonctions iteratives. *Journal de Mathematiques Pures et Appliquees [=Liouville's Journal]*, 10(3):101–108, 1884.

[89] P. Franklin. Relating to the real locus defined by the equation $x^y = y^x$. *American Mathematical Society*, 24:137, 1917.

[90] J. Friedman. *On the Convergence of Newton's Method*. PhD thesis, University of California, Berkeley, 1987.

[91] B. Frizel. The problem of defining the set of real numbers. *Bulletin of the American Mathematical Society [28-th regular meeting of the Chicago section of the AMS, University of Chicago, 28-30 December 1910].*, 17:296, 1910-1911.

[92] R. Garver. On the approximate solution of certain equations. *American Mathematical Monthly*, 39(8):476–478, Oct. 1932.

[93] D. Geisler. The Atlas of Tetration, 2004. http://www.tetration.org/.

[94] G.M. Ghunaym. Problem 4696. *School Science and Mathematics*, 99(1):54, Jan. 1999.

[95] R.A. Gibbs. Problem 4334. *School Science and Mathematics*, 92(5):292–293, May/Jun. 1992.

[96] J. Ginsburg. Iterated exponentials. *Scripta Mathematica II*, pages 292–293, 1945.

[97] J.W.L. Glaisher. A theorem in trigonometry. *Quarterly Journal of Pure and Applied Mathematics*, 15:151–157, 1878.

[98] F. Gobel and R.P. Nederpelt. The number of numerical outcomes of iterated powers. *American Mathematical Monthly*, 78:1097–1103, 1971.

[99] R.L. Goodstein. The equation $a^b = b^a$. *The Mathematical Gazette*, 28(279):76, May 1944.

[100] R. Greenberg. Problem e 1597 [1963, 757]. *American Mathematical Monthly*, 71(3):322, Mar. 1964.

[101] M.J.-H. Grillet. Les exponentielles successives d'Euler. *Journal de Mathematiques Pures et Appliquees [=Liousville's Journal]*, 10:233–241, 1985.

[102] Sir William Rowan Hamilton. On quaternions. *Proceedings of the Royal Irish Academy*, 3(1847):1–16, 11 Nov., 1844.

[103] Sir William Rowan Hamilton. On a new species of imaginary quantities connected with a theory of quaternions. *Proceedings of the Royal Irish Academy*, 2:424–434, 13 Nov., 1843.

[104] D. Hammer. Problem 171, parenthetical roots. *Mathematics and Computer Education*, 17(1):73, Winter 1983.

[105] M.E. Haruta. Newton's method on the complex exponential function. *Transactions of the American Mathematical Society*, 351(5):2499–2513, 1999.

[106] A. Hathaway. Quaternion space. *Transactions of the American Mathematical Society*, 1(3):46–59, Jan. 1902.

[107] A. Hausner. Algebraic number fields and the diophantine equation $m^n = n^m$. *American Mathematical Monthly*, 68:856–861, 1961.

[108] A. Hausner. Problem e 1474 [1961,573], the equation $m^{n^m} = n^{m^n}$. *American Mathematical Monthly*, 69(2):169, Feb. 1962.

[109] N.D. Hayes. The roots of the equation $x = (c \cdot exp)^n x$ and the cycles of the substitution $(x|ce^x)$. *Quarterly Journal of Mathematics (Oxford)*, 3(2):81–90, 1952.

[110] J. Van Heijenoort. *From Frege to Goedel: A Source Book in Mathematical Logic, 1879 - 1931. Section On Hilbert's construction of the real numbers, written by Wilhelm Ackermann (1928).* Harvard University Press, 1967.

[111] A.S. Hendler. Problem e 1144 [1954,711]. *American Mathematical Monthly*, 62(6):446, Jun./Jul. 1955.

[112] J.L. Hickman. Analysis of an exponential equation with ordinal variables. *Procceedings of the American Mathematical Society*, 61:105–111, 1976.

[113] A. Hoorfar and M. Hassani. Inequalities of the Lambert W function and hyperpower function. *Journal of Inequalities in pure and applied mathematics*, 9(2):51–56, 2008.

[114] S. Hurwitz. On the rational solutions of $m^n = n^m$, with m/nen. *American Mathematical Society*, 74:298–300, 1967.

[115] Jr. H.W. Labbers. Problem 420. *Nieuw Archief voor WisKunde*, 3(24):207–210, 1976.

[116] J.T. Varner III. Comparing $a^b = b^a$ using elementary calculus. *Two Year College Mathematics Journal*, 7(4):46, 1976.

[117] R.B. Israel. *Calculus the Maple Way.* Addison-Wesley, 1996.

[118] V.F. Ivanoff. Problem e 34 [1933,241]. *American Mathematical Monthly*, 44(2):101–106, Feb. 1934.

[119] S. J. J. G. Hagen. On the history of the extensions of the calculus. *Bulletin of the American Mathematical Society*, 6:381–390, 1899-00.

[120] D.J. Jeffrey. The multi-valued nature of inverse functions, 2001. Available online at: http://citeseer.ist.psu.edu/jeffrey01multivalued.html.

[121] H.W. Lenstra Jr. 3053. problem 566. *Nieuw Archief voor WisKunde*, 3(28):300–302, 1980.

[122] R.A. Knoebel. Exponentials reiterated. *American Mathematical Monthly*, 88:235–252, 1981.

[123] M. Kossler. On the zeros of analytic functions. *Proceedings of the London Mathematical Society*, 19(2):back pages for the January 13, 1921 Meeting, 1921.

[124] S.G. Krantz. *Handbook of Complex Variables*. Birkhäuser, Boston, MA, 1999.

[125] Y.S. Kupitz and H. Martini. On the equation $x^y = y^x$. *Elemente der Mathematik*, 55:95–101, 2000.

[126] L.J. Lander. Problem e 1124 [1954,423]. *American Mathematical Monthly*, 62(2):124–125, Feb. 1955.

[127] H. Länger. An elementary proof of the convergence of iterated exponentials. *Elemente der Mathematik*, 51:75–77, 1996.

[128] H. Langer. An elementary proof of the convergence of iterated exponentials. *Elemente Math.*, 51:75–77, 1996.

[129] J. Lense. Problem 3053. *American Mathematical Monthly*, 31:500–501, 1924.

[130] S. De Leo and P. Rotelli. A new definition of hypercomplex analyticity, January 1997. Dipartimento di Fisica, Università degli Studi Lecce INFN, Sezione di Lecce via Arnesano, CP 193, I-73100 Lecce, Italy, Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, via Belzoni 7, I-35131 Padova, Italy.

[131] S.L. Loney. *Plane Trigonometry, 5'th Edition*. Cambridge University Press, parts I and II, 1925.

[132] G.F. Lowerre. A logarithm problem and how it grew. *The Mathematics Teacher*, 72:227–229, 1979.

[133] S.V. Lüdkovsky and F. van Oystaeyen. Differentiable functions of quaternion variables. *Bulletin des Sciences Mathematiques*, 9(127):755–796, 2002.

[134] J. Van De Lune. Problem 899. *Nieuw Archief voor WisKunde*, 4(13):251–252, 1995.

[135] J. Macdonnell. Some critical points of the hyperpower function $x^{x^{\cdots^{x}}}$. *International Journal of Mathematical Education*, 20(2):297–305, 1989.

[136] A.J. Macintyre. Convergence of $i^{i^{\cdots}}$. *Procceedings of the American Mathematical Society*, 17:67, 1966.

[137] R.L. Mayes. Discovering relationships, logarithmic and exponential functions. *School Science and Mathematics*, 94(7):367–370, Nov. 1994.

[138] J. Milnor. Dynamics in one complex variable introductory lectures. Institute for Mathematical Sciences, SUNY, Stony Brook, New York, preprint 1990/5.

[139] M.C. Mitchelmore. A matter of definition. *American Mathematical Monthly*, 81:643–647, 1974.

[140] J. Morton. Graphical study of $x^x = y$. *The Mathematical Student Journal*, 9(2):5, Jan. 1962.

[141] E.J. Moulton. The real function defined by: $x^y = y^x$. *American Mathematical Monthly*, 23:233–237, 1916.

[142] R.P. Nederpelt. Elementary problems and solutions: E1903. *American Mathematical Monthly*, 79:395–396, 1972.

[143] D.J. Newman. Problem 4569 [1954,51]. *The Mathematical Student Journal*, 62(3):190–191, Mar. 1955.

[144] I. Niven. Which is larger, e^π or π^e? *Two Year College Mathematics Journal*, 3(2):13–15, 1972.

[145] S. Ocken. Convergence criteria for attracting cycles of Newton's method. *SIAM J. Appl. Math.*, 58(1):235–244, 1998.

[146] C.S. Ogilvy. Advanced problems and solutions: E853. *American Mathematical Monthly*, 56:555–556, 1949.

[147] L. O'Shaughnessy. Problem 433. *American Mathematical Monthly*, 26(1):37–39, Jan. 1989.

[148] F.D. Parker. Integrals of inverse functions. *The American Mathematical Monthly*, 62(6):439–440, 1955.

[149] H.-O. Peitgen and P.H. Richter. *The Beauty of Fractals*. Springer-Verlag, Berlin, Germany, 1986.

[150] J.J. Petersen and P. Hilton. $\exp(\pi) > \pi^e$? *Readers Reflections Column, Mathematics Teacher*, 74(7):501–502, Oct. 1981.

[151] H. Poritsky. Problem 2851 [1920,377]. *American Mathematical Monthly*, 29(3):132–133, Mar. 1922.

[152] S. Rabinowitz. Problem 191, exponential roots. *Mathematics and Computer Education*, 18(2):150–151, Spring 1984.

[153] W.R. Ransom. Problem e 3 [1932,489]. *American Mathematical Monthly*, 40(2):113, Feb. 1933.

[154] D. Redfern. *The Maple Handbook*. Springer-Verlag, 1996.

[155] D.L. Renfro. Exponential and logarithmic commutativity. *The Mathematics Teacher*, 91:275–362, 1998.

[156] J. Riordan. A note on Catalan parantheses. *American Mathematical Monthly*, 80:904–906, 1973.

[157] B. Ross and B.K. Sacheva. The inversion of Kepler's equation. *Int. J. Math. Educ. Sci. Technol.*, 16:558–560, 1985.

[158] R.R. Rowe. The mutuabola. *Journal of Recreational Mathematics*, 3(3):176–178, July 1970.

[159] S. Saks and A. Sygmund. *Analytic Functions*. Hafner Publishing Company, New York, 1952.

[160] H. E. Salzer. An elementary note on powers of quaternions. *American Mathematical Monthly*, 5(59):298–300, May 1952.

[161] D. Sato. Shorter notes: Algebraic solution of $x^y = y^x$, $(0 < x < y)$. *Procceedings of The American Mathematical Society*, 31:316, 1972.

[162] J. Scheffer. Problem 307. *American Mathematical Monthly*, 16(2):32, Feb. 1909.

[163] E.D. Schell. Problem e 640 [1944,472]. *American Mathematical Monthly*, 52(5):278–279, May 1945.

[164] D.L. Shell. *Convergence of Infinite Exponentials*. PhD thesis, University of Cincinnati, 1959.

[165] D.L. Shell. On the convergence of infinite exponentials. *Procceedings of The American Mathematical Society*, 13:678–681, 1962.

[166] T.W. Shilgalis. Graphical solution of the equation $a^b = b^a$. *The Mathematics Teacher*, 66:235, 1973.

[167] S.L. Siegel. Exponential equations. *Reader Reflections Columnm, Mathematics Teacher*, 100(4):238, Nov. 2006.

[168] D.J. Silverman. What's the limit? *Journal of Recreational Mathematics*, 4(2):144, Apr. 1971.

[169] H.L. Slobin. The solutions of $x^y = y^x$, $x, y > 0$, $x \neq y$ and their graphical representation. *American Mathematical Monthly*, 38:444–447, 1931.

[170] G.P. Speck. a^b versus b^a. *School Science and Mathematics*, 65(6):489–490, Jun. 1965.

[171] M. Spivak. *Calculus (English)*. Publish or Perish, Inc., Berkeley, California, 1980.

[172] V.M. Spunar. Problem 430. *American Mathematical Monthly*, 26(9):415, Nov. 1919.

[173] R.M. Sternheimer. On certain integers which are obtained by repeated exponentiation. *Journal of Recreational Mathematics*, 22(4):271–276, 1990.

[174] Daniel Drew (student). A recursion relation involving exponentials. *American Mathematical Monthly*, 56(9):660–664, Nov. 1949.

[175] Paul Heckbert (student). $|x|^y = |y|^x$. *The Mathematics Student Journal*, 22(4):4,7, Apr. 1975.

[176] A. Sudbery. Quaternionic analysis. *Mathematical Procceedings of the Cambridge Philosophical Society*, 85:199–225, 1979.

[177] W.R. Thomas. John Napier. *The Mathematical Gazette*, 19(234):192–205, Jun. 1935.

[178] W.J. Thron. Convergence of infinite exponentials with complex elements. *Procceedings of The American Mathematical Society*, 8:1040–1043, 1957.

[179] W.J. Thron. Convergence regions for continued fractions and other infinite processes. *American Mathematical Monthly*, 68:734–750, 1961.

[180] S.D. Turner. Under what conditions can a number equal its logarithm? (part 2). *School Science and Mathematics*, 28(240):376–379, Apr. 1928.

[181] S.D. Turner. Under what conditions can a number equal its logarithm? (part 1). *School Science and Mathematics*, 27(7):750–751, Oct. 1927.

[182] H.S. Uhler. On the numerical value of i^i. *American Mathematical Monthly*, 28(3):114–121, Mar. 1921.

[183] unknown. Problems, notes, 7-9. *American Mathematical Monthly*, 28(3):140–143, March 1921.

[184] unknown. Problems for solution. *American Mathematical Monthly*, 70(5):571–573, May 1963.

[185] G.T. Vickers. Experiments with infinite exponents. *Mathematical Spectrum*, 27(2):34, 1994/1995.

[186] G.T. Vickers. More about an infinite exponential. *Mathematical Spectrum*, 27(3):54–56, 1994/1995.

[187] J.M. De Villiers and P.N. Robinson. The interval of convergence and limiting functions of a hyperpower sequence. *American Mathematical Monthly*, 93:13–23, 1986.

[188] R. Voles. An exploration of hyperpower equations ${}^nx = {}^ny$. *The Mathematical Gazette*, 83(497):210–215, 1999.

[189] D. Wang and F. Zhao. The theory of smale's point estimation and its applications. *Journal of Computational and Applied Mathematics*, 60:253–269, 1995.

[190] M. Ward. Note on the iteration of functions of one variable. *Bulletin of the American Mathematical Society*, 40:688–690, 1934.

[191] M. Ward and F.B. Fuller. Continuous iteration of real functions. *Bulletin of the American Mathematical Society*, 42:383–386, 1936.

[192] G. Waters. problem 185. *Mathematics and Computer Education*, 18(1):69–70, Winter 1984.

[193] W.V. Webb. Rooting around for roots. *Mathematics and Computer Education*, 24(3):273, Fall 1990.

[194] Eric W. Weisstein. Conformal mapping, from mathworld–a wolfram web resource., 2011. http://mathworld.wolfram.com/ConformalMapping.html.

[195] Eric W. Weisstein. Contour winding number, from mathworld–a wolfram web resource., 2011. http://mathworld.wolfram.com/ContourWindingNumber.html.

[196] F. Woepcke. Note sur l' expression $(((a^a)^a)^{\cdots})^a$, et les fonctions inverses correspondantes. *Journal Fur Die Reine und Angewandte Mathematik [=Crelle's Journal]*, 42:83–90, 1851.

[197] V. Wood. Quaternions. *The Analyst*, 1(7):11–13, Jan. 1980.

[198] E.M. Wright. On a sequence defined by a non-linear recurrence formula. *Journal of the London Mathematical Society*, 20(1):68–73, 1945.

[199] E.M. Wright. Iteration of the exponential function. *Quarterly Journal of Mathematics (Oxford)*, 18:228–235, 1947.

[200] E.M. Wright. A class of representing functions. *Journal of the London Mathematical Society*, 29(1):63–71, 1954.

[201] A. P. Yefremov. Quaternions: Algebra, geometry and physical theories. *Hypercomplex Numbers in Geometry and Physics*, 1:104–119, 2004.

[202] S. Yeshurun. Reverse order exponentiation. *School Science and Mathematics*, 89(2):136–143, Feb. 1989.

[203] G. Zirkel. But does it converge? *Mathematics and Computer Education*, 18(2):153, Spring 1984.

[204] A.E. Dubinov and I.N. Galidakis. An explicit solution to Kepler's equation. *Physics of Particles and Nuclei. Letters*, 4(3):213–216, May 2007.

[205] I.N. Galidakis. Lambert's W function and convergence of infinite exponentials in the space of quaternions (corrigendum). *Complex Variables*, 52(4):351, Apr. 2007.

[206] I.N. Galidakis. Lambert's W function and convergence of infinite exponentials in the space of quaternions. *Complex Variables*, 51(12):1129–1152, Dec. 2006.

[207] I.N. Galidakis. On some applications of the generalized hyper-Lambert functions. *Complex Variables and Elliptic Equations*, 52(12):1101–1119, Dec. 2007.

[208] I.N. Galidakis. On solving the p-th complex auxiliary equation $f^{(p)}(z) = z$. *Complex Variables*, 50(13):977–997, Oct. 2005.

[209] I.N. Galidakis. On an application of Lambert's W function to infinite exponentials. *Complex Variables Theory Appl.*, 49(11):759–780, Sep. 2004.

[210] I.N. Galidakis and E.W. Weisstein. Power tower, from mathworld–a wolfram web resource., 2011. http://mathworld.wolfram.com/PowerTower.html.